Research Article ISSN: 2581-4672 Open Access

UPI JOURNAL OF MATHEMATICS AND BIOSTATISTICS

Journal Home Page: https://uniquepubinternational.com/upi-journals/upi-journal-of-mathematics-and-biostatistics-upi-jmb/

On a Class of Convexoid Type Operators

Luis Berbesi, Teodoro Lara*, Pedro Pena, Edgar Rosales

Departamento De FiSica Y Matematicas, Universidad De Los Andes, Trujillo, Venezuela.

Funding Source(s): NA

Editorial History:
Received : 15-02-2019, Accepted: 14-05-2019, Published: 16-05-2019

Correspondence to: Lara T. Departamento De FiSica Y Matematicas, Universidad De Los Andes, Trujillo, Venezuela.
Email: tlara@ula.ve

How to Cite: Berbesi L, Lara T, Pena P, Rosales E. On operator m-convex functions in Hilbert space. UPI Journal of Mathematics and Biostatistics 2019; 2(1): JMB14.

On A Class Of Convexoid Type Operators

Luis Berbesi, Teodoro Lara, Pedro Pena, Edgar Rosales

Abstract: In this research we introduce a class of convexoid type operators, the so called analytically convexoid operators, defined on Hilbert spaces. Consequences and examples of this definition are exhibited; spectrum of this class consisting of one single point is studied in detail.

Key words: Numerical range, nilpotent operator, convexoid operator, complex Hilbert-space, completely reduced operator.

1. Introduction

In this research we introduce a class of convexoid type operators, that will be named as analytically convexoid operators, this class contains that of convexoid operators, which makes its study an interest issue. We describe completely those analytically convexoid operators whose spectra is a singleton.

Along this work $\mathbb{H}$ will denote a complex infinite dimensional Hilbert space, $\mathbb{C}$ the set of complex numbers and $L(\mathbb{H})$ the class of bounded linear operators defined on $\mathbb{H}$. Many classes of operators T on Hilbert spaces are convexoid. Let $T \in L(\mathbb{H})$; the incoming concepts are in the available literature, for instance in [6], T is called *self-adjoint* if $T = T^*$, where T^* is its adjoint; *Normal* if $TT^* = T^*T$; *quasinormal* when $T(T^*T) = (T^*T)T$; *subnormal* if it has a normal extension to a normal operator N on a larger Hilbert space $\mathbb{K} \supset \mathbb{H}$ with $Tx = Nx$, $x \in \mathbb{H}$ arbitrary; *hyponormal* if $T^*T \geq TT^*$; *normaloid* if $||T|| = r_\sigma(T)$, with $r_\sigma(T) = \sup\{|\lambda|; \lambda \in \sigma(T)\}$ the spectral radius of T; *transaloid* if $\mu I - T$ is normaloid, for any $\mu \in \mathbb{C}$. The numerical range $W(T)$ of $T \in L(\mathbb{H})$ is [6]

$$W(T) = \{\langle Tx, x\rangle : ||x|| = 1\}.$$

The Toeplitz-Hausdorff Theorem [6] and references therein, states that $W(T)$ is a convex subset of the complex plane and $co(\sigma(T)) \subset \overline{W(T)}$, where $co(\sigma(T))$ means convex hull of $\sigma(T)$.

Definition 1. *([5, 11]) An operator $T \in L(\mathbb{H})$ is called* convexoid operator *if $\overline{W(T)} = co(\sigma(T))$.*

Examples of this type of operators are the projection, self-adjoint, unitary, isometry, partial isometry, normal, quasinormal, subnormal, hyponormal and transaloid [6].

A bounded linear operator $T \in L(X)$, defined on a complex infinite dimensional Banach space X, is said to be nilpotent, if there exists $m \in N$ such that $T^m = 0$.

If $T \in L(X)$ satisfying $r_\sigma(T) = 0$, is said to be a quasi-nilpotent. If T es nilpotent, then T es quasi-nilpotent.

2. Results

In this section we present our results, but before doing that is necessary to recall few known results in order to make a little easier the reading and understanding of this research. Many of these results are discussed, in a broader and more detailed way, in [3, 6, 7, 10].

Theorem 2. *For T being a linear transformation from a two-dimensional Hilbert space into itself the following takes place.*

(1) *If T has different eigenvalues λ_1 and λ_2, with x_1 and x_2 respective unitary eigenvectors, then $W(T)$ is a closed elliptical disc, with foci in λ_1 and λ_2; if $r = |\langle x_1, x_2\rangle|$ y $\delta = \sqrt{1 - r^2}$, then the length of the minor axis is*

$$\frac{r|\lambda_1 - \lambda_2|}{\delta},$$

while the length of the major axis is

$$\frac{|\lambda_1 - \lambda_2|}{\delta}.$$

(2) *If T has a single eigenvalue λ, then $W(T)$ is the disc with center in λ and radius*

$$\tfrac{1}{2}||T - \lambda I||.$$

We recall that for X being a vector space and T a linear operator in X, a subspace M of X is called *T-invariant* or *invariant under T* if $T(M) \subseteq M$. If M_1 and M_2 are independent vector subspaces of X with $X = M_1 \oplus M_2$; T is said to be *completely reduced* by the pair (M_1, M_2) if M_1 and M_2 are invariant under T.

Theorem 3. *Let X be a complex Banach space and $T \in L(X)$. Suppose that T is completely reduced by the pair (M_1, M_2), and T_i is the restriction of T to the subspace M_i, $i = 1, 2$. Then*

(1) *$\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$.*
(2) *$W(T) = co\{W(T_1) \cup W(T_2)\}$.*

The incoming example from [6] exhibits a convexoid operator.

Example 1. *Let c_n be a dense succession of the unitary disc of the complex plane, that is,*

$$\overline{\{c_n : n \in \mathbb{N}\}} = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Consider the operator $N : \ell^2 \to \ell^2$ given by

$$Nx = (c_1 x_1, c_2 x_2, c_3 x_3, \ldots),$$

where $x = (x_1, x_2, x_3, \ldots)$. Then N is a normal operator, with $\overline{\{c_n : n \in \mathbb{N}\}} = \sigma(N)$.

Let T be the operator with matrix representation

$$\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix},$$

where $M : \mathbb{C}^2 \to \mathbb{C}^2$ is the operator given by $M(x,y) = (0,x)$, and N is the normal operator described above. By the Theorem 3,

$$\begin{aligned} \sigma(T) &= \sigma(M) \cup \sigma(N) \\ &= \{0\} \cup \{z \in \mathbb{C} : |z| \le 1\} \\ &= \{z \in \mathbb{C} : |z| \le 1\}. \end{aligned}$$

Again, by the Theorem 3,

$$\begin{aligned} W(T) &= co(W(M) \cup W(N)) \\ &= co(\{z \in \mathbb{C} : |z| \le \tfrac{1}{2}\} \cup \{z \in \mathbb{C} : |z| \le 1\}) \\ &= \{z \in \mathbb{C} : |z| \le 1\}. \end{aligned}$$

Since $co(\sigma(T)) = \overline{W(T)}$, then T is a convexoid operator.

Remark 4. *If $\mathbb{H} \neq \{0\}$, then $W(T) \neq \emptyset$ and $\sigma(T) \subseteq \overline{W(T)}$.*

Let $\mathbb{H}_{nc}(\sigma(T))$ be the set of all analytic functions defined in an open neighborhood of $\sigma(T)$, such that f is non-constant in each component of its domain. For each function $f \in \mathbb{H}_{nc}(\sigma(T))$ we are able to define $f(T)$ just by means of the classsical functional calculus.

Our first result runs as follows

Lemma 5. *Suppose that $T \in L(\mathbb{H})$, the following statements are equivalent.*

(1) *T is convexoid and $\sigma(T) = \{\lambda\}$.*
(2) *$W(T) = \{\lambda\}$.*
(3) *$T = \lambda I$.*

Proof. (1) $\Rightarrow$ (2) T convexoid implies $co(\sigma(T)) = \overline{W(T)}$. Clearly, $\overline{W(T)} = \{\lambda\}$, hence $W(T) = \{\lambda\}$.
(2) $\Rightarrow$ (3) If $W(T) = \{\lambda\}$ we let $x \in \mathbb{H}$; $x = 0$ means $Tx = \lambda 0 = \lambda I(0) = \lambda I x$, while for $x \neq 0$, $\langle (T - \lambda I)x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle = \lambda||x||^2 - \lambda||x||^2 = 0$. Therefore $\langle (T - \lambda I)x, x\rangle = 0$, for all $x \in \mathbb{H}$ and since $\mathbb{H}$ is complex Hilbert, $T - \lambda I = 0$, so $T = \lambda I$.
(3) $\Rightarrow$ (1) Suppose $T = \lambda I$, so $W(T) = W(\lambda I) = \{\lambda\}$, and $\emptyset \neq \sigma(T) \subseteq \{\lambda\} = \overline{W(T)}$, therefore, $\sigma(T) = \{\lambda\} = co(\sigma(T))$. Thus $\overline{W(T)} = \{\lambda\} = co(\sigma(T))$, in other words T is convexoid and $\sigma(T) = \{\lambda\}$. □

Definition 6. *An operator $T \in L(\mathbb{H})$ is said to be* analytically convexoid *if there exists $f \in \mathbb{H}_{nc}(\sigma(T))$ such that $f(T)$ is convexoid. In particular, if $h \in \mathbb{C}[x]$ is a non constant polynomial such that $h(T)$ is convexoid, T is called* algebraically convexoid.

It is worth to point out that definitions of convexoid and algebraic convexoid operators are not equivalent as the incoming example shows.

Example 2. *If $T \in L(\mathbb{H})$ is convexoid, then T is algebraically convexoid since by choosing $h(t) = t$, then $h(T) = T$ which is is convexoid. Now if T is algebraically convexoid operator it is not necessarily convexoid. Actually for $T : \mathbb{C}^2 \to \mathbb{C}^2$ given as $T(x, y) = (0, x)$, $T^2(x, y) = (0, 0)$, or T^2 is convexoid, and consequently T is algebraically convexoid. On the other hand, the Spectral Mapping Theorem ensures that $\sigma(T) = \{0\}$ and by Theorem 2,*

$$W(T) = \left\{ z \in \mathbb{C} : ||z|| \leq \frac{1}{2} \right\},$$

thus $co(\sigma(T)) = \{0\} \neq \overline{W(T)}$ and T is not a convexoid operator.

The incoming remarks show up

Remark 7. (1) *If T is nilpotent and convexoid, then $\sigma(T) = \{0\}$ and $co(\sigma(T)) = \{0\}$. Since T is convexoid we have $\overline{W(T)} = \{0\}$ and consequently, $T = 0$. In other words, the only operator T nilpotent and convexoid is $T = 0$. Thus, if $T \in L(\mathbb{H})$ is nilpotent and $T \neq 0$, it is not convexoid.*

(2) *Every nilpotent operator is algebraically convexoid, since $T^n = 0$, for some $n \in \mathbb{N}$ natural and the null operator is convexoid. On the other hand, if $T \neq 0$ and nilpotent, then T is not convexoid.*

The main result of our research is the following

Theorem 8. *Suppose that $\sigma(T) = \{\lambda\}$ and T is analytically convexoid, then $T = \lambda I + N$, where N is a nilpotent operator.*

Proof. Suppose that $f(T)$ is convexoid, for some $f \in \mathbb{H}_{nc}(\sigma(T))$; from the Spectral Mapping Theorem,

$$\sigma(f(T)) = f(\sigma(T)) = \{f(\lambda)\},$$

and the Lemma 5, $f(T) = f(\lambda)I$.

Now, let $g(t) = f(\lambda) - f(t)$, clearly $g \in \mathbb{H}_{nc}(\sigma(T))$, $g(\lambda) = 0$, and g has only a finite number of zeros in $\sigma(T)$ [4]. Let $\{\lambda, t_1, \ldots, t_n\}$ be the set of all zeros of g, where $\lambda \neq t_i$; $t_i \neq t_j$, for all $i \neq j$, and t_i with multiplicity $n_i \in \mathbb{N}$. Hence

$$g(t) = \mu(t - \lambda)^m \prod_{i=1}^{n} (t_i - t)^{n_i} h(t),$$

where $h(t)$ has no zeros in $\sigma(T)$. Since $g(T) = f(\lambda)I - f(T) = 0$,

$$0 = g(T) = \mu(T - \lambda I)^m \prod_{i=1}^{n} (t_i I - T)^{n_i} h(T),$$

and operators $t_i I - T$, $h(T)$ invertible ones. But then $(T - \lambda I)^m = 0$ and if $N = T - \lambda I$, N becomes a nilpotent operator, with $T = \lambda I + N$ as desired. □

Corollary 9. *If $T \in L(\mathbb{H})$ is quasi-nilpotent and analytically convexoid, then T is nilpotent.*

Proof. $\sigma(T) = \{0\}$ since it is quasi-nilpotente and by Theorem 8, there is a nilpotent operator N such that $T = 0I + N = N$. □

Corollary 10. *If $T \in L(\mathbb{H})$ is algebraically convexoid and quasi-nilpotent, then T is nilpotent.*

Remark 11. *If $T \in L(\mathbb{H}$, then $Ext(\sigma(T)) \subset \partial\sigma(T)$, where $Ext(\sigma(T))$ denote the extreme points of $\sigma(T)$ and $\partial\sigma(T)$ is the boundary of $\sigma(T)$. Since $\sigma(T)$ is compact, $co(\sigma(T))$ is compact and convex. Now, by the Krein-Milman theorem in finite-dimension,*

$$\begin{aligned} co(\sigma(T)) &= \overline{co(Ext(\sigma(T)))} \\ &\subset co(\partial\sigma(T)) \\ &\subset co(\sigma_a(T)) \\ &\subset co(\sigma(T)), \end{aligned}$$

Therefore, $co(\sigma_a(T)) = co(\sigma(T))$.

Remark 12. *T es convexoid if and only if $co(\sigma_a(T)) = \overline{W(T)}$.*

One important fact that deserves to be mentioned here is that property for an operator being convexoid is no preserved under perturbation, actually if $N : \mathbb{C}^2 \to \mathbb{C}^2$ is given as $N(x, y) = (0, x)$ then N is nilpotent, $\sigma(N) = \{0\}$ and $W(N) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$. Now we pick $\alpha \in \mathbb{C}$ and $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(x, y) = \alpha(x, y)$, then T is convexoid, $\sigma(T) = W(T)$, and $\sigma(T + N) = \sigma(T) = \{\alpha\}$, but

$$W(T + N) = co[W(T) \cup W(N)] = \alpha + \left\{z \in \mathbb{C} : |z| \leq \frac{1}{2}\right\};$$

in other words, $T + N$ is not convexoid. In a similar manner, convexoid property is not inherited under restrictions on an invariant closed subspace neither. Indeed, if M, N and T are as in example 1, then T is convexoid, nevertheless, the operator M (the restriction of T to $\mathbb{H} = \mathbb{C}^2$) is not convexoid, since $\sigma(M) = \{0\}$ and $W(M) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$.

3. Conclusion

We have introduced the class of the so called analytically convexoid operators; as an illustration, we described completely those whose spectra is a singleton. Relations with other types of operators, like quasi-nilpotent and algebraically convexoid operators are stated as well.

4. Conflicts of Interest

The author(s) report(s) no conflicts of interest(s) and the author along are responsible for the content and writing of the paper.

5. References

1. Aiena P, Guillen J, Pena P. A unifying approach to weyl type theorems for Banach Operators. Integral Equations and Operator Theory 2013; 77(3): 371-384.
2. Aiena P, Guillen J. Weyl's theorem for perturbations of paranormal operator. Proceedings of the American Mathematical Society 2007; 135(8): 2443-2451.
3. Bachman G, Narici L. Functional Analysis, Academic Press, New York, 1966.
4. Conway J. Functions of one complex variable, 2nd Edition. Springer-Verlag, New York, USA, 1978.
5. Furuta T. Certain convexoid operators. Proceedings of the Japan Academy, 47 (Suppl 1), 1971, pp. 888-893.
6. Furuta T. Invitation to linear operators. Taylor & Francis, London, England, 2002.
7. Goldberg M, Tadmon E. On the numerical radius and its applications, Elsevier Science Publishing, New York, USA, 1982.
8. Halmos P. Hilbert space problem book, Springer-Verlag, New York, 1982.
9. Heuser H. Functional analysis. Marcel Dekker, New York, USA, 1982.
10. Kreyszig E. Introductory functional analysis with applications, Wiley Classic Library Edition, Ontario, Canada, 1989.
11. Salah M. The numerical range of linear operators. Filomat 2008; 22(2): 1-8.
12. Taylor A, Lay D. Introduction to functional analysis, Wiley Classic Library Edition, New York, USA, 1980.