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Abstract: In this research the m-convexity for functions on time scales, defined on subsets of real linear spaces, is introduced. We state results similar to those for this type of convexity on real numbers; a sandwich like theorem and some inequalities of the Jensen type are also exhibited.

Key words: m-convex function, time scale, epigraph, sandwich theorem, Jensen type inequalities.

1. Introduction

The theory of time scales was introduced by S. Hilger in his PhD thesis [2]. The calculus and applications of dynamic derivatives on time scales provide an unification and an extension of traditional differential and difference equations. The most important ones include the dynamic equations, which include both differential equations and difference equations [3]. In this research we introduce the concept of m-convex functions in the context of time scale on $\mathbb{R}$, topological and algebraic properties are set and proved for this type of functions same as some Jensen like inequalities. It is worth to point out that the results and concepts established here come from similar ones for the case of real functions defined subsets of $\mathbb{R}$. The concept of m-convex function $(0 \le m \le 1)$ is given in ([4, 11]), more recently in [7] and references therein. The incoming notations are from [2] and [3]. A time scale (or measure chain) is any nonempty closed subset $\mathbb{T} \subseteq \mathbb{R}$. Along this paper $\mathbb{T}$ will denote a time scale and, I and interval of $\mathbb{R}$, $I_{\mathbb{T}} = I \cap \mathbb{T}$ a time scale subset. Unless other issue is specified, $I_{\mathbb{T}} \neq \emptyset$. Our first definition, inspired in the corresponding for real functions ([4, 5, 6, 7] among others), runs as follows,

Definition 1. A function $f : I_{\mathbb{T}} \to \mathbb{R}$ is called m-convex on $I_{\mathbb{T}}$, $0 \le m \le 1$, if

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y).$$

for all $x, y \in I_{\mathbb{T}}$ and all $t \in [0,1]$ provided that $tx + m(1-t)y \in I_{\mathbb{T}}$. The function f is strictly m-convex on $I_{\mathbb{T}}$ if the above inequality is strict for distinct $x, y \in I_{\mathbb{T}}$ and $t \in [0,1]$. The function f is m-concave (respectively, m-strictly concave) on $I_{\mathbb{T}}$, if $-f$ is m-convex (respectively, m-strictly convex).

Note that if f is m-convex on $I_{\mathbb{T}}$ and $0 \in I_{\mathbb{T}}$, then for each $x \in I_{\mathbb{T}}$ and $t \in [0,1]$, $f(tx) \le tf(x)$, whenever $tx \in I_{\mathbb{T}}$; functions enjoying this property are said to be starshaped in $I_{\mathbb{T}}$; moreover, $f(0) \le 0$. In a similar way, $I_{\mathbb{T}}$ is called starshaped set if for any x in it and $t \in [0,1]$, the element $tx \in I_{\mathbb{T}}$ as well.

The foregoing couple of examples show that the concepts of convexity and m-convexity of functions on time scales, neither are equivalent nor one of them implies the other.

Example 1. Let $I = [0,2]$, $\mathbb{T} = [0,1]$ and $f : \mathbb{T} \to \mathbb{R}$ given as $f(x) = x + 1$, clearly f is a convex function on $\mathbb{T}$ [3, Theorem 3.4], but it is not m-convex on this domain since $f(0) > 0$ for any $m \in (0,1)$.

Example 2. Now we set $I = [0,1]$, $\mathbb{T} = [0, \frac{1}{2}]$ and $f : \mathbb{T} \to \mathbb{R}$ defined by $f(x) = -x^2 - 1$; f is $\frac{1}{2}$- convex on $\mathbb{T}$. Notice that f is a concave function, and because of Theorem 3.4 in [3], there is not convex function $\tilde{f} : [0,1] \to \mathbb{R}$ such that $\tilde{f} = f$ on $[0, \frac{1}{2}]$; in other words, f is not a convex function on $\mathbb{T}$.

Incoming result is similar to Theorem 1.1 from [4].

Proposition 2. *If $f : I_{\mathbb{T}} \to \mathbb{R}$ is m-convex, $0 \leq m < 1$, and $I_{\mathbb{T}}$ is starshaped, then f is n-convex for any $0 \leq n \leq m$.*

Proof. Note that, for any $x, y \in I_{\mathbb{T}}$ and $0 \leq t \leq 1$, if $tx + m(1-t)y \in I_{\mathbb{T}}$,

$$tx + n(1-t)y = tx + m(1-t)\left(\frac{n}{m}\right) y \in I_{\mathbb{T}}$$

because of starshapedness of $I_{\mathbb{T}}$. Hence,

$$\begin{aligned} f\left(tx + n(1-t)y\right) &= f\left(tx + m(1-t)\left(\frac{n}{m}\right) y\right) \\ &\leq tf(x) + m(1-t) f\left(\frac{n}{m} y\right) \\ &\leq tf(x) + m(1-t)\frac{n}{m} f(y) \\ &= tf(x) + n(1-t) f(y), \end{aligned}$$

the last inequality by hypothesis on f. □

Definition 3. ([10, 11, 12])The epigraph of a function $f : I_{\mathbb{T}} \to \mathbb{R}$, $epif$, is the set

$$epif = \{(x,y) \in I_{\mathbb{T}} \times \mathbb{R} : y \geq f(x)\}.$$

An immediate consequence of this definition [12] is,

Proposition 4. *The function $f : I_{\mathbb{T}} \to \mathbb{R}$ is m-convex if and only if $epif$, is m-convex, that is, (x_1, y_1), $(x_2, y_2) \in epif$ implies $t(x_1, y_1) + m(1-t)(x_2, y_2) \in epif$, whenever $tx_1 + m(1-t)(x_2 \in I_{\mathbb{T}}$ for any $0 \leq t \leq 1$.*

Proof. Let f be m-convex and (x_1, y_1), $(x_2, y_2) \in epif$; if $tx_1 + m(1-t)x_2 \in I_{\mathbb{T}}$ for any $t \in [0,1]$, the element $\big(tx_1 + m(1-t)x_2, ty_1 + m(1-t)y_2\big)$ is in $epif$ if $f(tx_1 + m(1-t)x_2) \leq ty_1 + m(1-t)y_2$, which is true because of hypothesis. In the other direction, let x_1, $x_2 \in I_{\mathbb{T}}$ such that $f(x_1) \leq y_1$ and $f(x_2) \leq y_2$, hence (x_1, y_1) and (x_2, y_2) are in $epif$. If $tx_1 + m(1-t)x_2 \in I_{\mathbb{T}}$, then

$$f(tx_1 + m(1-t)x_2) \leq ty_1 + m(1-t)y_2.$$

Now if $f(x_1), f(x_2)$ are both finite, by choosing $y_1 = f(x_1)$ and $y_2 = f(x_2)$ we are done. In case $f(x_1) = -\infty$ or $f(x_2) = -\infty$ is enough to set $y_1 \to -\infty$, $y_2 \to -\infty$ respectively. □

Foregoing Proposition characterizes m-convex functions in time scale in terms of their epigraph, which is a well known result for m-convex real functions defined on real intervals, as it may be noticed proof in the context of time scale is very similar provided the condition of points being in $I_{\mathbb{T}}$.

2. Results and Discussion

This Section deals essentially with algebraic results involving the class of m-convex functions on time scale defined on subset $I_{\mathbb{T}}$ as before. The incoming two results run in a similar manner to ones given in [7, 10].

Proposition 5. *Let $f, g : I_{\mathbb{T}} \to \mathbb{R}$ be functions such that f is m_1-convex and g is m_2-convex, $m_1 \leq m_2$, then $f + \alpha g$ is m_1-convex for any constant $\alpha \geq 0$.*

Proof. We easily may prove that being g an m_2-convex function and α as in the hypothesis, the function αg is m_2-convex and consequently m_1-convex (Proposition 2), now for any $x, y \in I_{\mathbb{T}}$, $0 \leq t \leq 1$; and assuming $tx + m_1(1-t)y \in I_{\mathbb{T}}$,

$$\begin{aligned}(f+\alpha g)(tx+m_1(1-t)y) &= f(tx+m_1(1-t)y) + \alpha g(tx+m_1(1-t)y)\\ &\leq tf(x) + m_1(1-t)f(y) + t(\alpha g)(x) + m_1(1-t)(\alpha g)(y)\\ &= t(f+\alpha g)(x) + m_1(1-t)(f+\alpha g)(y).\end{aligned}$$

□

Proposition 6. *If $f, g : I_{\mathbb{T}} \to \mathbb{R}$ are both m-convex, nonnegative and similarly ordered on $I_{\mathbb{T}}$, i.e. for arbitrary $x, y \in I_{\mathbb{T}}$,*

$$\big(f(x) - f(y)\big)\big(g(x) - g(y)\big) \geq 0, \tag{2.1}$$

then the product function fg is m-convex as well.

Proof. From (2.1) and arbitraries $x, y \in I_{\mathbb{T}}$,

$$f(y)g(x) + f(x)g(y) \leq f(x)g(x) + f(y)g(y), \tag{2.2}$$

hence for $x, y \in I_{\mathbb{T}}$ as before, $t \in [0,1]$ and $tx + m(1-t)y \in I_{\mathbb{T}}$,

$$\begin{aligned}(fg)(tx+m(1-t)y) &= f(tx+m(1-t)y)g(tx+m(1-t)y)\\ &\leq \big[tf(x)+m(1-t)f(y)\big]\big[tg(x)+m(1-t)g(y)\big]\\ \text{(because of (2.2))} &\leq [t^2+mt(1-t)]f(x)g(x)\\ &\quad +[m^2(1-t)^2+mt(1-t)]f(y)g(y)\\ \text{(since } t+m(1-t)\leq 1\text{))} &\leq tf(x)g(x)+m(1-t)f(y)g(y).\end{aligned}$$

□

The following two results are similar to Proposition 2.1 and 2.2, respectively, from [6]. We omit the proof.

Proposition 7. *If f_1, $f_2 : I_{\mathbb{T}} \to \mathbb{R}$ are m-convex functions, then the function given by $f(x) = \max_{x \in I_{\mathbb{T}}}\{f_1(x), f_2(x)\}$ is also m-convex.*

Proposition 8. *For a given sequence $f_n : I_{\mathbb{T}} \to \mathbb{R}$ of m-convex functions pointwise converging to a function f on $I_{\mathbb{T}}$, the limit f is m-convex as well.*

We finish this Section with a result of a sandwich type similar to the one for the real case stated in [8].

Theorem 9. *Let $f : I_{\mathbb{T}} \to \mathbb{R}$ be an m-convex function, $0 < m < 1$, then there exists a convex function $h : I_{\mathbb{T}} \to \mathbb{R}$ such that*

$$f(x) \leq h(x) \leq mf\left(\frac{x}{m}\right),\, x \in I_{\mathbb{T}},\ \textit{whenever}\ \frac{x}{m} \in I_{\mathbb{T}}.$$

Proof. Because of the m-convexity of f

$$f(tx + (1-t)y) \leq tmf\left(\frac{x}{m}\right) + (1-t)mf\left(\frac{y}{m}\right),$$

for arbitrary $x, y \in I_{\mathbb{T}}, t \in [0,1]$ and, of course, $tx, (1-t)y, \frac{x}{m}, \frac{y}{m} \in I_{\mathbb{T}}$. Next, a function g is defined by

$$g(x) = mf\left(\frac{x}{m}\right),\ x \in I_{\mathbb{T}}, \text{whenever } \frac{x}{m} \in D_{\mathbb{T}}.$$

Hence,

$$f\left(tx + (1-t)y\right) \leq tg(x) + (1-t)g(y),$$

apply sandwich theorem for convexity on time scale [9] and conclude the existence of convex function $h : I_{\mathbb{T}} \to \mathbb{R}$ with

$$f(x) \leq h(x) \leq g(x), \text{ for } x, \frac{x}{m} \in I_{\mathbb{T}}.$$

In other words,

$$f(x) \leq h(x) \leq f\left(\frac{x}{m}\right).$$

□

3. More Results: Inequalities of Jensen Type

Here we exhibit some inequalities of Jensen type following ideas from [4, 6] and references therein.

Theorem 10. *Let $f : I_{\mathbb{T}} \to \mathbb{R}$ be an m-convex function, $0 \leq m < 1$, then for any finite collection of real numbers $t_i > 0$, and $x_i \in I_{\mathbb{T}}$; $i = 1, \cdots, n$, n arbitrary, such that*

$$\frac{1}{T_n}\sum_{i=1}^{n} t_i m^{i-1} x_i \in I_{\mathbb{T}} \ \textit{with}\ T_n = \sum_{i=1}^{n} t_i, \tag{3.1}$$

the inequality

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i m^{i-1} x_i\right) \leq \frac{1}{T_n}\sum_{i=1}^{n} t_i m^{i-1} f(x_i) \tag{3.2}$$

holds.

Proof. The proof runs by mathematical induction on n, if $n = 2$, we use hypothesis to obtain (3.2), in other words, $T_2 = t_1 + t_2$ and

$$f\left(\frac{1}{T_2}[t_1 x_1 + mt_2 x_2]\right) \leq \frac{t_1}{T_2} f(x_1) + \frac{mt_2}{T_2} f(x_2).$$

Assume now (3.2) takes place for $n-1$, hence

$$f\Big(\frac{1}{T_{n-1}}\sum_{i=1}^{n-1} t_i m^{i-1} x_i\Big) \leq \frac{1}{T_{n-1}}\sum_{i=1}^{n-1} t\ m^{\ -1} f(x_i).$$

On the other hand,

$$\frac{1}{T_n}\sum_{i=1}^{n} t_i m^{i-1} x_i = \frac{t_1}{T_n} x_1 + m\left(1-\frac{t_1}{T_n}\right)\sum_{i=2}^{n}\frac{t_i}{T_n - t_1} m^{i-2} x_i,$$

but $\sum_{i=2}^{n}\frac{t_i}{T_n-t_1} m^{i-2} x_i \in I_{\mathbb{T}}$ because of (3.1). Hence

$$\begin{aligned}
f\Big(\frac{1}{T_n}\sum_{i=1}^{n} t_i m^{i-1} x_i\Big) &\leq \frac{t_1}{T_n} f(x_1) + m\Big(1-\frac{t_1}{T_n}\Big) f\Big(\sum_{i=2}^{n}\frac{t_i}{T_n-t_1} m^{i-2} x_i\Big) \\
\text{(because of inductive hyp.)} &\leq \frac{t_1}{T_n} f(x_1) + m\left(1-\frac{t_1}{T_n}\right)\left[\frac{1}{T_n-t_1}\sum_{i=2}^{n} t_i m^{i-2} f(x_i)\right] \\
&= \frac{1}{T_n}\sum_{i=1}^{n} t_i m^{i-1} f(x_i).
\end{aligned}$$

□

Theorem 11. *Let* $t_1, \ldots, t_n > 0$, n *any integer number, and* $T_n = \sum_{i=1}^{n} t_i$ *as in the foregoing Theorem. If* $f : I_{\mathbb{T}} \to \mathbb{R}$ *is an* m*-convex function, with* $m \in (0,1)$, *for all* $x_1, \ldots, x_n \in I_{\mathbb{T}}$ *with* $\frac{1}{T_n}\sum_{i=1}^{n} t_i m^{i-1} x_i$ *and* $\frac{x_i}{m^{n-i}}$ *both in* $I_{\mathbb{T}}$, *then*

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \leq \frac{1}{T_n}\sum_{i=1}^{n} m^{n-i} t_i f\left(\frac{x_i}{m^{n-i}}\right). \tag{3.3}$$

Proof. The proof also goes by induction on n; for $n = 2$ inequality (3.3) is easily verified. Suppose now that (3.3) is satisfied for $n-1$, this is

$$f\left(\frac{1}{T_{n-1}}\sum_{i=1}^{n-1} t_i x_i\right) \leq \frac{1}{T_{n-1}}\sum_{i=1}^{n-1} m^{n-1-i} t_i f\left(\frac{x_i}{m^{n-1-i}}\right),$$

hence

$$f\left(\frac{1}{T_{n-1}}\sum_{i=1}^{n-1} t_i \frac{x_i}{m}\right) \leq \frac{1}{T_{n-1}}\sum_{i=1}^{n-1} m^{n-1-i} t_i f\left(\frac{x_i}{m^{n-i}}\right), \tag{3.4}$$

also

$$\begin{aligned} f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) &= f\Big(\frac{t_n}{T_n}x_n + m\frac{T_{n-1}}{T_n}\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\frac{x_i}{m}\Big) \\ (\text{since } f \text{ is } m\text{-convex}) &\leq \frac{t_n}{T_n}f(x_n) + m\frac{T_{n-1}}{T_n}f\Big(\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\frac{x_i}{m}\Big) \\ (\text{because of } (3.4)) &\leq \frac{t_n}{T_n}f(x_n) + \frac{m}{T_n}\sum_{i=1}^{n-1} m^{n-1-i} t_i f\left(\frac{x_i}{m^{n-i}}\right) \\ &= \frac{1}{T_n}\sum_{i=1}^{n} m^{n-i} t_i f\left(\frac{x_i}{m^{n-i}}\right). \end{aligned}$$

□

4. Conclusion

In the present research we introduced the concept of m-convex function in the context of time scale, properties, examples and Jensen inequalities of the this type of function are shown in a similar manner to the case of m-convex function in the real case, it is worth to mention that our investigation, we believe, open the the door to future research of more properties in this area.

5. Conflicts of Interest

The author(s) report(s) no conflicts of interest(s) and the author along are responsible for content and writing of the paper.

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