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Basic Concepts of the Theory of Ample Probability

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Abstract

Employing dynamic or causal structuring of events and their representation in the form of named sets, we construct a probability function called ample probability for such events and develop elements of an axiomatic ample probability theory. We show (Theorem 4.2) that axioms that characterize ample probability are consistent and independent. It is also proved that in the limit, i.e., when all named sets representing events are single named sets, the axiom system for ample probability becomes Kolmogorov's axiom system for conventional probability (Theorem 4.1). Using the introduced axiomatic system of ample probability, we study properties of ample probability. In addition, we show how ample probability works by means of structuration of events for calculating their probabilities.

Key words: Probability, Axiom, Transition, Independence, Consistency.

1. Introduction

The mathematical methods of probability first arose in 1560s (though not published until 100 years later due to his fame as a heretic) in the investigations first of Gerolamo Cardano, who was a famous Italian polymath. For quite a while, the term chance was used in the sense of probability and probability theory was called Geometry or Doctrine of Chances from Cardano's *Book on Games of Chance* [18].

However, the birth of probability theory is usually ascribed to the correspondence of Pierre de Fermat and Blaise Pascal in 1650s. Between 1651 and 1654 Blaise Pascal attended a pair of probability problems investigating the fair division of the stake in an interrupted game of chance. The solutions to those problems inspired an innovative vision of mathematics to him. Pascal forecasted the organization of the probability domain as a new

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mathematical discipline which he named *geometrie des chances* [50].

The scientific community adhered to Pascal's manifesto and since then worked to discover the abstract properties of probability. In this venue, Christiaan Huygens gave the earliest comprehensive treatment of the subject around 1657 [40]. Authors achieved significant theoretical results while the application of the probability and the statistics progressively expanded in science and engineering.

In the 20th century, Kolmogorov built an axiomatic probability theory [43], which is now the universally accepted mathematical model in this area. He defined the logic of the probability domain in a form symmetrical to Euler's geometry, yet the *geometry of chance* was not born in spite of the fundamental Kolmogorov's contribution [36]. Moreover, various other quantities that have nothing to do with probability do satisfy Kolmogorov's axioms, and thus are interpretations of it in a strict sense: normalized mass, length, area, volume, and indeed anything that falls under the scope of measure theory, an abstract mathematical theory that generalizes such quantities.

As a result, mathematicians are not inclined to recognize any *geometry of chance* even if there is no logical incongruity in the axiomatic theory. In addition, no construct has been unanimously accepted as the official 'theory of probability' so far.

Why we have this uncertain intellectual situation?

1.1. Double Challenge

Mathematicians argued on probability at the abstract level and achieved significant results, but the relations of probability with the world out there appeared to be ever more complex by time passing. We confine ourselves to a couple of remarks.

1.1.1. James Bernoulli first investigated the relationships between the theoretical value of probability and the practical experience proving that the frequency method and the classical method are consistent with one another in his book Ars Conjectandi in 1713 [6].

Several authors worked out and improved the Law of Large Numbers (LLN). Kolmogorov also felt the need of presenting LLN in the last chapter of his seminal work and this turns out to be surprising [43]. In principle, pure mathematicians do not care how a theory should be tested or employed in the living environment. A theorist can even create a construct that has no application, yet his construct is valid providing it is consistent. By contrast, LLN and the ways how one can test probability in the world play central role for theorists.

1.1.2. During the nineteenth century probability calculus progressively, infiltrated new sectors including quantum mechanics. The scientific community has become progressively aware of the prismatic character of probability and discovered that basic notions are not so intuitive as the basic concepts are in any mathematical construct. Nowadays thinkers still tackle such questions as:

- i) What does probability mean?
- ii) What does probability qualify?
- iii) What quality or qualities does probability quantify?

Basically these queries center on the probabilistic phenomena occurring in the world out there, rather than in purely mathematical topics. They are the sign of the importance of phenomenology at the theoretical level demonstrating that probability is not only a mathematical concept but also a scientific notion.

At this point we face two alternative conclusions: either experts in probability are not sufficiently professional, or the experimental side of the probability theory has a weight which is unusual in mathematics.

We share the second way since we notice that Pascal's program did not launch a single challenge but a double challenge which may be presented this manner.

Scientists inquired deterministic phenomena for millennia, while random phenomena waited in the background. When Pascal meant to inaugurate a new *geometry of chances* he automatically drew the attention of scientists to physical effects that had been systematically ignored until then. The concept of 'chance' was a novelty for experimentalists besides logicians. They were forced to move the eyes towards the indeterministic phenomenology of the world which is complementary to the deterministic phenomenology. Determinism and indeterminism look like the animal and plant kingdoms which include all the living beings. Therefore it was not so easy for experts to discover the second side of the world and the probability that measures it.

1.2. Axiomatics for Probability

Kolmogorov's axiomatics for probability theory was a milestone in its development [43]. This theory has been set up using purely mathematical criteria and has not been disproved so far, thus it cannot be rejected. However, there is a necessity in a new frame, which would illustrate in more detail the random mechanisms emerging in the world. The multifold aspects of probability should be explained by means of a scheme that is more specific than the existing scheme. We clarify ourselves using the following example for the history of physics.

Newton and Lagrange have developed two exhaustive descriptions of mechanics [60]. Physical phenomena are the same in both approaches but the two mathematical constructions are very dissimilar. Lagrange's mechanics starts with the function L as the difference of the kinetic energy T from the potential energy V

$$L = T - V$$

The Lagrangian mechanics uses a set of generalized coordinates q_1, q_2,...q_n for the conservative system with n degrees of freedom; hence under hypotheses which we overlook here, the equations of motion of the system have this compact form,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \qquad j = 1, 2, \ldots n$$

When the Lagrangian L does not depend on a certain coordinate q_k then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} = 0 \qquad \Rightarrow \qquad \frac{\partial L}{\partial \dot{q}_k} = C$$

Where C is a constant and we say that $\partial L / \partial \dot{q}_k$ is a conserved quantity. Conservation of energy arises when the Lagrangian has no explicit time dependence $\partial L / \partial t = 0$ on the basis of this elegant expression

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}$$

The Lagrangian mechanics turns out to be extremely simple in principle, showing that the most varied consequences "may be derived from one radical formula" as Hamilton wrote. However Langrange's mechanics cannot be understood without the help of the classical mechanics. Newton's mechanics begins with the following laws of motion which spell out the concept of force

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

Next, the work W is defined as equal to the line integral of the force F along the path C

$$W = \int_C F\mathrm{d}s$$

This allows to obtain kinetic energy

$$T = \int_C F\mathrm{d}s = \int_C v\mathrm{d}(mv) = \int_C \mathrm{d}(mv^2/2) = mv^2/2$$

and the potential energy with near Earth gravity under some constraints

$$V = \int_C F\mathrm{d}s = \int_{t_1}^{t_2} Fv\mathrm{d}t$$

Lagrangian fromalism takes view which could be defined as global and axiomatic; it opens the doors to the Hamiltonian mechanics which is applicable to much of fundamental theoretical physics. On the other hand, Newtonian mechanics has a local 'cause/effect' perspective and proves to be more itemized. We could say classical mechanics is more 'elementary' but Lagrange's formulas cannot be explained without it.

On our opinion, something like should be set up in the probability domain. Kolmogorovian frame proves to be axiomatic, concise and elegant. We need to add up a more powerful theoretical frame capable of spelling out the phenomena typical for the indeterministic realm. Kolmogorov expresses the probability properties using the minimum representation possible. The new itemized frame should use more equations. It should employ models that are perhaps less agile than the axiomatic models but are suitable to clarify the basic questions about probability that are waiting an answer since three centuries. In any case, it will clarify queries from i to iii using the mathematical language and not philosophical argumentations.

In mechanics, Newton came first and Lagrange later. The second frame did not raise dispute thanks to its predecessor. Unfortunately, mathematicians were captivated by Pascal's *geometrie of chances* and at first searched the most abstract formalism. A probability theorist did not conform to the right order and as a consequence certain confusion has emerged inside the probability sector. The goal of this paper is to set up a more specific formalism and use it to construct a more accurate probability theory.

1.3. Events and Probability

The current literature agrees on the idea that probability qualifies random events. It means that the concept of an event is basic for probability theory and thus, to have a relevant probability theory, we need an adequate model of an event.

In the conventional probability theory, events are identified with their outcomes. For instance, in contemporary textbooks and treatises on probability, experiments are considered having a space Ω of possible outcomes and then subsets of Ω are called events (cf., for example, [64]). In essence, all approaches represent events as static entities while this does not exactly reflect reality, which is permanently changing. As the ancient Geek philosopher Heraclites said, all things forever flow and change. However in some cases, changes in the initial conditions are not essentially different and it is possible to ignore these differences.

Statisticians, who are closer to real life than probabilists, encountered this problem much earlier when they found that the way in which data are collected is critical while data constitute the initial conditions for statistical inference (cf., for example, [46]). Data collected well can yield powerful insights and discoveries. Data collected poorly can yield very misleading results.

Collecting data from a sample, statisticians try to predict properties or behavior of the whole population. However, these properties are changing and it is necessary to take these changes into account. Now there is a big problem in science that landmark experiments often cannot be replicated [1, 30, 48, 55]. One of the reasons is that it is impossible to repeat an experiment in exactly the same conditions and it is necessary to take into account changes in the initial conditions. However, the conventional probability theory does not take into account these changes. Indeed, in their classical works, von Mises and Kolmogorov consider a complex of initial conditions and treat events as what happens or what can happen when the given initial conditions are realized [43, 63]. As a result, initial conditions are supposed to be the same for all considered events.

In contrast to this, in ample probability theory, the foundations of which are developed in this paper, events are dynamically structured including initial (beginning) conditions and concluding outcomes. This structuration of events better reflects scientific conception of events providing efficient means to study and utilize probability taking into account changing initial conditions.

In this paper, we build a mathematical model of dynamically structured events in the form of a named set. This model allows elaboration of a theory of ample probability using means and constructions from named set theory – a novel mathematical theory, which unifies set theory, logic, fuzzy set theory, multiset theory, category theory and many other mathematical disciplines. Ample probability is probability of dynamically structured events. Based on this mathematical representation of events, we construct an axiomatic foundation of ample probability theory. An important property of our axiomatics is that when we reduce events to their outcomes, we obtain the classical Kolmogorov's axiomatics.

The paper has the following structure. In Section 2, we introduce the informal notion of a structured event explaining necessity of this stratification. In Section 3, we impart elements of named set theory necessary for the development of ample probability theory. In Section 4, we build an axiomatics for ample probability exploring its relations to the conventional probability (Theorem 4.1) and proving consistency of the axiomatics and independence of its axioms (Theorem 4.2). As the constructed axiomatics directly corresponds to the classical Kolmogorov axiomatics, it is natural to treat the constructed axiomatic theory as the theory of standard ample probability. In Section 5, we study properties of ample probability in the axiomatic context, while in Section 6 we show how ample probability works through structuration of events.

2. Dynamic Structure of Events

As a dynamic essence, an event is a change and this change goes in space and time. This shows that an event has three groups of characteristics:

1. Temporal characteristics reflect time of changes.
2. Spatial characteristics reflect where the changes take place.
3. Substantial characteristics reflect what changes take place in the event and how these changes go on.

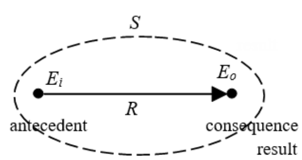
Different scientific fields make emphasis on those groups that are relevant to the studied systems and processes.

2.1. The first two groups of characteristics are studied in the theory of concurrent computations. Duration of event is important for efficiency of computation and communication and thus, is the base for computational complexity. As a result, a model of true concurrency should support abstractions for events whose occurrence partially overlap in time. Thus, the possibility of observed event simultaneity is not merely a Boolean proposition, but rather a continuum: events A and B may overlap entirely, partially, or not at all. This is done in the Event-Action-Process model (EAP) of concurrent processes and systems [14 – 16]. EAP considers the meaning of an event from three perspectives: the change in the process that engaged in the event, the change in the environment in which the event occurred, and the change in the relationship between a process and its environment when an event occurs.

2.2. The third group of characteristics plays an important role in ample probability theory. In it, an event begins with certain initial conditions including input, which is processed during a certain period of time, and at the end emits the outcome, e.g., the result. Hence the event has a structure which encompasses three base components: the antecedent, a dynamical evolution and the final outcome [56 - 58]. As a result, an event is represented by the following mathematical structure S including the entities E_i and E_o, and the relation R between them

$$S = (E_i , R, E_o) \qquad (2.1)$$

We can visualize the whole S and its elements using this graph



The event's core is the dynamics R that receives E_i and brings forth E_o. It may be said the event S occurs by means of R. For example, ninety balls are the input of the lottery event, the drawing machine of the lottery is R and the number released is the output E_o.

Physics and the theory of abstract automata consider events in state (phase) space. Event as dynamic entity in a state (phase) space of a finite automaton has three components (cf., for example, [11])

(Start state, transition, Final state)

Thus, a formal representation of an event in the theory of abstract automata is

$$(St_{in}, tr, St_{out})$$

In a similar way, an event in physics also has three components (cf., for example, [22])

(Start state, operator, Final state)

Thus, a formal representation of an event in physics is

(St_{in}, op, St_{out})

In the causality theory, an event always includes its cause and effect, which implies the triadic structure of an event [20, 55].

In addition, von Foerster writes that events are representations of relations [62], while relations are special cases of named sets [12].

An event in psychology and decision-making also has three components,

(Initial conditions, action, Outcome)

An event in logic has the following components,

(Axioms, deduction, Theorems)

or

(Assumptions, induction, Conclusions)

An event in the theory of computation has the following components,

(Input, computation, Output)

All this brings us to the conclusion that an event in probability theory must also have three components,

(Initial conditions, experiment, Outcome)

As in the structural model (2.1), this structure is mathematically modeled by a name set allowing using the named set theory [12] for the development of a mathematical theory of probability for structured events.

2.3. Now it is possible to classify the cause/effect description of S in the following terms. The antecedents E_i and the mechanism R are the known components of an event. This gives us three mutually exclusive possibilities (2.2):

- **a.** The structural elements E_i and R establish that the result E_o will certainly occur.
- **b.** The structural elements E_i and R establish that the result E_o will certainly not occur.
- **c.** The structural elements E_i and R do not systematically establish the result E_o will occur.

Conditions **a** and **b** are typical of the *deterministic events*, in particular **a** describes the *certain* events and **b** the *impossible* events. The event **c** whose resulting behavior is not entirely determined by the antecedents is called *random* or *indeterministic*. In detail, given E_i and R sometimes the outcome E_o is brought forth; sometimes it is not done. That is to say, *sometimes S occurs and sometimes does not occur in the world*. Adopting the Bernouilli scheme, S will be called the *successful* event and the alternative *not-S* is the *unsuccess*.

2.4. The exhaustive classification **a, b** and **c** enables us to qualify both deterministic and indeterministic events using *the probability P(S) that quantifies the degree of existence of S.*

The probability $P(S)$ calculate how likely the event S can occur when the antecedents E_i and the operation R are given.

(2.3)

From (2.2) we derive the following values for $P(S)$

$P(S)=1$ (or) $P(S)=0$ when S is a deterministic event

$1> P(S) >0$ when S is an indeterministic event (2.4)

Probability calculus crosses multiple fields, and involves different contexts, scopes etc. For philosophers, probability has a variety of meanings; it may be intended as possibility, logical implication, feasibility, prediction and so forth. From the present perspective $P(S)$ *qualifies solely the degree of existence of S*, and thus there is a direct logical correspondence between $P(S)$ and the relative frequency $F(S)$, when the first is theoretical and the second empirical

$$P \leftrightarrow F \qquad (2.5)$$

The Law of Large Numbers corroborates this conclusion.

2.5. Let us examine the relation of the present frame with usual formalism used in the probability domain.
The axiomatic theory and other probability constructs assume the event S is a subset of the event space Ω

$$S \in \Omega \qquad (2.6)$$

Because S is a subset, S is a generic collection of elementary parts. Formula (2.1) specifies the parts pertaining to S and is consistent with (2.6).

The conventional approach identifies the event with the result

$$S = (E_o) \qquad (2.7)$$

Definition **c** of (2.2) holds that the event S is random because the elements E_i and R do not systematically establish the result E_o. This definition entails that the event *not-S* occurs in alternative to S when *not-S* has the following structure

$$Not\text{-}S = (E_i, not\text{-}E_o; R) \qquad (2.8)$$

Formulas (2.1) and (2.8) prove that the outputs E_o and *not-*E_o identify the overall events S and *not-S* respectively. Hence (2.7) is an agile way to present the structure of the random event. This conclusion fits with the introduction of the present paper which holds that the axiomatic theory is concise as the Lagrangian mechanics whereas the present frame tends to justify all the details in similitude with the Newtonian work.

Actually, in the conventional probability theory, events are structured because they have some initial conditions [43]. However, as these conditions are always the same for all considered events, it is possible to identify these events with their outcomes.

It is possible to consider *deterministic events* when given initial conditions result in a unique outcome and *nondeterministic events* when given initial conditions allow different outcomes.

2.6. Dynamic structure of events allows taking into account such important parameter as time. Namely, considering an event $S = (E_i , R, E_o)$, it is possible to include time t_b of the beginning of the event S in the component E_i and time t_e of the end of the event S in the component E_o . In many cases, it is important to know not only the probability of an event S but the probability of an event S at time t, or more exactly, the probability that an event S starts at time t, or the probability that an event S ends at time t. For instance, for people and governments, it is important to know the probability that a financial crisis will be next year but not the probability that a financial crisis will happen sometimes. For people who live near an ocean and authorities, it is much more important to know the probability that a tsunami will come next month in comparison with the probability that a tsunami will happen sometimes.

We will call events that contain temporal characteristics in their components *time-determined*. Probability of time-determined events is called *time sensitive*. In the opposite situation, i.e., when time is not taken into account, i.e., it is not included into the structure of events, probability is called *time free*.

Note that ample probability can belong to any of these two types.

2.7. Dynamic structure of events allows defining composition of events when the outcome of the first event forms (is included in) the initial conditions for the second event. This composition provides more adequate and thorough representation of conditional probability and independence of events. Informally, it is natural to define these concepts in the following way.

Let us consider two events S_1 and S_2 . Conditional probability is one of the basic concepts of the conventional probability theory [35, 43, 59, 61]. Some researchers even take conditional probabilities as basic and defining unconditional probabilities in terms of them. Hájek explains that conditional probability is in fact a pre-theoretic notion, and thus cannot be taken to be a purely technical [36].

In ample probability theory, there are three types of conditional probability because events have duration.

Definition 2.1. The *a posteriori conditional probability* of an event S_2 with respect to an event S_1 is the probability of S_2 under the condition that the event S_1ended before the event S_2 started.

Definition 2.2. The *concurrent conditional probability* of an event S_2 with respect to an event S_1 is the probability of S_2 under the condition that the event S_1 is happening when the event S_2 started.

Definition 2.3. The *a priori conditional probability* of an event S_2 with respect to an event S_1 is the probability of S_2 under the condition that the event S_1can happen after the event S_2.

Note that in the conventional probability theory, there is only one type of conditional probability.

Conditional probability brings us to the concept of probabilistic independence. In general, we have two classes of probabilistic independence: ordered independence and symmetric independence. At first, we define ordered independence.

Definition 2.4. An event S_2 is *independent* from an event S_1 if the conditional probability of S_2 coincides with the unconditional probability of S_2.

Taking this definition as the first level of the concept formation, we see that on the next level, three types of conditional probability give us three types of independence.

Definition 2.5. a) An event S_2 is *a posteriori independent* from an event S_1 if the a posteriori conditional probability of S_2 coincides with the unconditional probability of S_2.

b) An event S_2 is *a priori independent* from an event S_1 if the a priori conditional probability of S_2 coincides with the unconditional probability of S_2.

c) An event S_2 is *concurrently independent* from an event S_1 if the a concurrent conditional probability of S_2 coincides with the unconditional probability of S_2.

Ordered independence ordered is used to define symmetric independence.

Definition 2.6. Two events S_1 and S_2 are *independent* if each of them is independent from the other one.

Naturally, three types of (ordered) independence give us three types of symmetric independence.

Definition 2.7. a) Two events S_1 and S_2 are *a posteriori independent* if each of them is a posteriori independent from the other one.

b) An event S_2 is *a priori independent* if each of them is a priori independent from the other one.

c) An event S_2 is *concurrently independent* if each of them is concurrently independent from the other one.

To formalize ample probability, we employ named set theory [12], elements of which are presented in the next section.

3. Elements of the Theory of Named Sets

Here we describe algebras of named sets and study their properties used for the development of ample probability theory.

3.1. There are three primary types of named sets and fundamental triads – basic, bidirectional and cyclic fundamental triads (named sets). Here we consider only basic named sets (basic fundamental triads).

Definition 3.1. A *basic named set* also called a *basic fundamental triad* is a triad **X** = (X, f, N) with the following visual (graphic) representation:

$$X \xrightarrow{\quad f \quad} N \qquad (3.1)$$

In this triad **X** = (X, f, N), components X and N are two objects and f is a correspondence (e.g., a binary relation) from X to N. With respect to **X**, X is called the *support* of **X** and denoted by S(**X**), N is called the *component of names* (*reflector*) or *set of names* of **X** and denoted by N(**X**) , and f is called the *naming correspondence* (*reflection*) of **X** and denoted by r(**X**) . Note that in the named set **X**, components X and N are not always sets, while f is not necessarily a mapping or a function even if X and N are sets. For instance, X and N are sets of words and f is an algorithm.

Example 3.1 The standard example is a basic named set, in which X consists of people, N consists of their names and f is the correspondence between people and their names.

Example 3.2 Another example is a basic named set, in which X consists of things, N consists of their names and f is the correspondence between things and their names [24].

As we can see the structure of an event described in the previous section is also a basic named set (basic fundamental triad).

To make our exposition simpler, we use the term *named set* instead of *basic named set* in what follows.

A named set is a relatively new concept. Thus, to understand this concept and the role of named sets in mathematics and beyond, let us thoroughly analyze the concept of a function as well as related concepts of correspondence and binary relation.

Function and *set* are the most fundamental structures of the whole mathematics. Indeed, in our days, many perceive mathematics as the study of functions or relations on sets. This demonstrates the pivotal role of function in mathematics. For instance, Froelich *et al*. [32] write "the concept of function is probably the single most important idea in mathematics."

Moreover, some axiomatizations of mathematics foundations are based not on the concept of set but on the concept of a function [64-66].

Due to this fact, the concept of function is central for mathematics education. For instance, in the *Curriculum and Evaluation Standards for School Mathematics* (1989), the National Council of Teachers of Mathematics stated that "one of the central themes of mathematics is the study of patterns and functions."

In addition, *function* is the most basic concept in science in general and in physics in particular. For instance, Manin [47] writes that *correlation functions* are the main observable quantities in Quantum Field Theory. Besides, operators are also functions while contemporary quantum physics uses operators as its main mathematical tool [67].

It is interesting that although functions were used at least from the time of ancient Babylonian mathematics, mathematicians started to use the term *function* only in the 17th century, while the concept itself was defined only in the middle of the 18th century. This term was informally utilized by Leibniz [44] when he explained that lines in a figure perform some function. Euler [29] was the first mathematician who defined *function*:

A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.

This understanding of function prevailed for almost two centuries but when people came to the necessity to delegate computations of functions to computers, they found that this definition was insufficient. The same conclusion also came from the rigorous mathematical analysis of the concept of a function [7]. As a result, the modern definition of a function as it is, for example, presented in the book [37] states:

A function from X to Y is a triple (triad) (X, f, Y) *where* $f \subseteq X\times Y$ *is a binary relation such that for each* $x \in X$ *there is exactly one* $y \in Y$ *such that* $(x, y) \in f$.

In a similar way, Chinn and Steenrod [21] write

A function f consists of three things: a set X called the domain of f, a set Y called the range of f, and a rule that assigns to each element of X a corresponding element from Y.

As a function is a special case of a binary relation, these definitions of a function stem from the earlier approach to the definition of a correspondence or binary relation, which was introduced by Bourbaki [7]:

A correspondence between sets A and B is a triple (A, G, B) *where G is a graph of the correspondence, i.e., a set of pairs, for which the projection* $\mathrm{pr}_1\, G \subseteq A$ *and the projection* $\mathrm{pr}_2\, G \subseteq B$.

Thus, we can see that a function is a named set (fundamental triad) of the form (X, f, Y) where the support X is the domain, the reflector Y is the range of the function and the reflection f is the correspondence between elements from X and Y defined by this function.

The same is true for a correspondence or a binary relation. Namely, it is a named set (fundamental triad) of the form (A, G, B) where the support A is the domain, the reflector B is the codomain of the correspondence and the reflection G is the connection between elements from X and Y defined by this correspondence.

Interestingly Lipschutz [45] also defines binary relations as fundamental triads (named sets).

A relation R consists of a set A, a set B and an open sentence $\mathrm{P}(x, y)$, *in which* $\mathrm{P}(a, b)$ *is either true or false for any ordered pair* (a, b) *that belongs to* $A \times B$.

Note that in the Lipschutz's definition the reflection of relation as a named set is not the graph of a function but an open sentence P(x, y). So, we see that functions and correspondences (binary relations) are special cases of named sets.

However, the majority of mathematicians assume sets being the basic structures in mathematics. So, let us contemplate how sets are related to named sets. From the first glance, sets look as more primitive than named sets. However, this is only an illusion. First, there are many named sets, such as morphisms in categories, which are not set-theoretical, i.e., which are independent of sets. Second, named sets have axiomatics, which are independent of sets and set theory. Third, and this is the most important, sets are particular cases of named sets. Indeed, sets always have names. As Poincare [53] explains, without a name, no object exists in science or in mathematics. Consequently, all elements of a set are connected to the name of this set, e.g., all elements of the set X are connected to X, i.e., they have a common name "an element from the set X." And this means that any set implicitly is a singlenamed set, i.e., a special case of named sets.

Moreover, in the 20th century, a variety of set generalizations were introduced, studied and applied. They include fuzzy sets [69], multisets [42], interval sets [49], set-valued sets [19], Boolean valued sets [4], intuitionistic fuzzy sets [2] and rough sets [51], to mention but a few. All of them are special cases of named sets or fundamental triads.

We show this for the most popular of them - fuzzy sets and multisets. According to an informal definition,

A multiset is a collection that is like a set but can include identical or indistinguishable elements.

According to an informal definition,

A multiset M on a set S, which is called the domain of M, is a pair (S, ρ) where is an equivalence relation on S.

It is assumed that equivalent elements are of the same sort, while elements in different equivalence classes are of different sorts. Thus, taking the set of sorts, or types, I, it would be natural to represent the multiset M on the set S by the named set **M** = (S, t, I) where $t(x)$ is the sort (type) of the element x from S.

Another even more popular generalization of sets is *fuzzy set*.

A fuzzy set A in a set U is defined by the pair (A, μ_A) where $\mu_A: U \to [0, 1]$ is a function.

Consequently, a fuzzy set A in a set U is the named set **A** = $(U, \mu_A, [0,1])$, where $\mu_A: U \to [0,1]$ is a membership function of A and $\mu_A(x)$ is the degree of membership in A of the element $x \in U$.

In addition, there is a diversity of other mathematical systems, which are particular cases of named sets or fundamental triads. To name but a few, operators, variables, binary relations, homomorphisms and isomorphisms of algebraic systems, cardinal and ordinal numbers, multinumbers [12], homeomorphisms of topological spaces, vectors, vector fields, tensor fields, morphisms in categories [33], graphs and hypergraphs [5], fiber spaces, topological and fiber bundles [41] are special cases of named sets or fundamental triads. Besides, it is demonstrated that named sets form unified foundations of mathematics comprising set theory, category theory, theory of algorithms and logic [8, 10].

In essence, named set is the most fundamental and at the same time, elementary structure in mathematics and beyond [9, 13].

It is necessary to make a distinction between triples and triads. A *triple* is any set with three elements, while a *triad* is a system of three connected elements (components). It is worthy of note that mathematicians introduced the concept of a triple in an abstract category [3]. In essence, such a triple is a triad that consists of three fundamental triads and thus is a triad of the second order [12]. Understanding of the complex nature of the *categorical triple* compelled to mathematicians to change the name of this structure and now it is always called a *monad* [68]. Interestingly, this shows connection between fundamental triads and Leibniz monads [44]. However, in this case, a monad consists of triads and not the other way around.

3.2. There are different classes of named sets. However, the most popular in mathematics class of named sets consists of set-theoretical named sets, which are utilized here for the development of ample probability theory.

Definition 3.2 [12]**.** A named set **X** = (X, r, I) is called *set-theoretical* if X and I are sets and r is a subset of the Cartesian product $X\times I$, i.e., in the traditional terminology, r is a binary relation between sets X and I and we have $r \subseteq X\times I$.

We see that a set-theoretical named set structurally coincides with a correspondence (binary relation) in the sense of Bourbaki [7]. Consequently, a bulk of named sets in mathematics is set-theoretical. Examples set-theoretical named sets are functions, operators, variables, binary relations, homomorphisms, homeomorphisms, vectors, vector fields, tensor fields, graphs and hypergraphs, fiber spaces, topological and fiber bundles, ordinary sets, fuzzy sets and multisets to mention but a few.

Examples of not set-theoretical named sets are: algorithmic named sets, in which the reflection (naming correspondence) is an algorithm; and morphisms in categories, which are categorical named sets.

In what follows we consider only set-theoretical named sets as mathematical models of events. However, establishment of an algorithmic probability theory will demand employment of algorithmic named sets, while development of a probability theory in categories will need categorical named sets.

3.3. The natural class of named sets adequate for representation of structured events is formed by binormalized named sets, which form the conceptual mathematical foundation for ample probability theory.

Definition 3.3 [12]**.** A named set **X** = (X, r, I) is called *binormalized* if it does not have empty names and uninterpreted elements in the support.

In terms of events, it means that any outcome of an event has some initial conditions (cause) and initial conditions (cause) of an event bring about some outcomes.

Definition 3.4 [12]. a) A named set $\Lambda = (\varnothing, \varnothing, \varnothing)$ is called the *empty named set*.

b) A named set **U**$\Lambda_X = (X, \varnothing, \varnothing)$ is called an *above empty named set* or *unnamed set*.

c) A named set **L**$\Lambda_I = (\varnothing, \varnothing, I)$ is called a *below empty named set* or *non-interpreted named set*.

d) A named set **M**$\Lambda_X = (X, \varnothing, I)$ is called a *middle empty named set* or *unnamed set*.

There are many examples of such named sets.

Example 3.3. Let us consider the named set **SP** the support of which consists of subatomic particles, such as proton, neutron or electron. In the 18th century physics, all these particles were unnamed because they were not discovered at that time. Physicists did not know about their existence. Thus, **SP** was an above empty named set because these particles did not have names in the 18th century, but it was not empty named set because these particles existed.

Example 3.4. Let us consider the named set **SA** of atoms the support of which consists of some particles called atoms and described by a system of properties, while the set of names consists of one word *atom*. In ancient Greece, two Attic philosophers Democritus (460-370 BCE) and Leucippus (fifth century BCE) suggested that the whole universe consists of a void and a very large number of invisible and indivisible particles that were called *atoms*. Thus, even at that time, the named set **SA** was not empty from above. However, there were no particles with properties ascribed to atoms by these philosophers and their followers. Consequently, at that time, the named set **SA** was empty from below.

Lemma 3.1. The empty named set Λ is binormalized, while above empty named sets, middle empty named sets and above empty named sets, which do not coincide with Λ, are not binormalized.

Lemma 3.2. If r(**X**) = ∅, then **X** belongs to one of the four types of empty named sets.

A unit named set is an important structure for ample probability theory.

Definition 3.5 [12]. A named set **X** is called a *unit named set* if each of the sets S(**X**), r(**X**) and N(**X**) consists of one element.

Example 3.5. Any morphism in a category is a unit named set.

Example 3.6. Any edge in a graph is a unit named set.

Example 3.7. Any transition of a finite automaton is a unit named set.

Proposition 3.1. A unit named set is binormalized.

A named subset is an important structure for ample probability theory.

Definition 3.6 [12]. A named set **Y** = (Y, q, J) is called a *named subset* of a named set **X** = (X, r, I) if $Y \subseteq X$, $J \subseteq I$ and $q = r \cap (Y \times J) = r|_{(Y,J)}$.

It is denoted by **Y** ⊆ **X**.

Note that a named subset of a binormalized named set is not always binormalized.

Example 3.8. Let us consider the binormalized named set **U** = (U = {0}, f = {(0, 1), (0, 2)}, E = {1, 2}). Then its named subset **X** = ({0}, ∅, ∅) is not binormalized.

However, for ample probability theory, we specify the concept of named subset introducing binormalized named subsets.

Definition 3.7. A named subset **Y** of a named set **X** is called a *binormalized named subset* of **X** if **Y** is binormalized.

It is denoted by **Y** $\subseteq_{\mathbf{B}}$ **X**.

Lemma 3.2. The empty named set Λ is a binormalized named subset of any named set.

Definition 3.8 [12]. A named subset **Y** = (Y, q, J) of a named set **X** = (X, r, I) is called a *unit named subset* if it is a unit named set.

Proposition 3.1 implies the following result.

Corollary 3.1. A unit named subset of a binormalized named set is a binormalized named subset.

3.4. Probability theory provides a double calculus, which consists of two parts: event calculus and the corresponding probability calculus.

a) In the conventional probability theory, the calculus of events is based on set theory employing set-theoretical operations, while the calculus of probabilities is based on operations with set-theoretical functions.

b) To build a probability theory for dynamically structured events, the calculus of events is based on named set theory employing operations with named sets, while the calculus of probabilities is based on operations with functions on named sets.

Therefore we will need the following operations with named sets for the ample probability theory.

Definition 3.9 [12]. A named set **Y** = (Y, q, J) is called the *union* of named sets **X** = (X, r, I) and **Z** = (Z, p, K) if $Y = X \cup Z$, $J = I \cup K$, and $q = r \cup p$.

It is denoted by **Y** = **X** ∪ **Z**.

Example 3.9. Let us consider named sets **X** = (X, r, I) and **Z** = (Z, p, K) with the following components

$$X = \{x, y, z\},$$
$$Z = \{y, z, u\},$$
$$I = \{a, b\},$$
$$K = \{c\},$$
$$r = \{(x, a), (y, b)\},$$
$$p = \{(z, c), (u, c)\}$$

To build **Y** = **X** ∪ **Z**, we take

$$Y = X \cup Z = \{x, y, z, u\},$$
$$J = I \cup K = \{a, b, c\},$$
$$q = r \cup p = \{(x, a), (y, b), (u, c), (z, c)\}$$

In such a way, we have

Y = (Y, q, J)

Example 3.10. Join of relations in relational databases is union of the corresponding named sets [25, 28].

Remark 3.1. In a similar way, it is possible to build the union of any number of named sets.

Union of named sets has many properties similar to properties of union of sets [12]. We present these properties as well as properties of other named set constructions in the form of propositions and lemmas, the majority of which is proved in the book [12].

Proposition 3.2. (*Idempotent Law*) **X** ∪ **X** = **X** for any named set **X**.

Proposition 3.3. (*Identity Law*) **X** ∪ Λ = Λ ∪ **X** = **X** for any named set **X**.

Proposition 3.4. (*Commutative Law*) **X** ∪ **Y** = **Y** ∪ **X** for any named sets **X** and **Y**.

Proposition 3.5 (*Associative Law*). **X** ∪ (**Y** ∪ **Z**) = (**X** ∪ **Y**) ∪ **Z** for any named sets **X**, **Y**, and **Z**.

Proposition 3.6 (*Monotone Law*). If **X** ⊆ **Z**, then **Y** ∪ **X** ⊆ **Y** ∪ **Z** for any named sets **X**, **Y**, and **Z**.

Corollary 3.2. If **X** , **Y** ⊆ **Z**, then **Y** ∪ **X** ⊆ **Z** for any named sets **X**, **Y**, and **Z**., i.e., the union of named subsets of a named set **Z** is a named subset of **Z**.

Corollary 3.3. If **Y** ⊆ **X**, then **X** ∪ **Y** = **X** for any named sets **X** and **Y**.

Proposition 3.7. The union of binormalized named sets is binormalized.

Propositions 3.1 and 3.7 imply the following result.

Corollary 3.4. The union of unit named sets is binormalized.

Proposition 3.8. Any binormalized named set is the union of its entire unit named subsets.

In the theory of named sets, there is also the concept of the intersection of set-theoretical named sets

Definition 3.10 [12]. A named set **Y** = (Y, b, J) is called the *intersection* of named sets **X** = (X, a, I) and **Z** = (Z, c, K) if $Y = X \cap Z$, $J = I \cap K$, and $b = a \cap c$.

It is denoted by **Y** = **X** $\cap$ **Z**.

Example 3.11. Intersection of multisets is an example of the intersection of the named sets that represent these multisets (cf., for example, [34]).

Remark 3.2. In a similar way, it is possible to build the intersection of any number of named sets.

Union of named sets has many properties similar to properties of union of sets [12].

Proposition 3.9. (*Annihilation Law*) **X** $\cap \Lambda = \Lambda \cap$ **X** $= \Lambda$ for any named set **X**.

Proposition 3.10. (*Idempotent Law*) **X** $\cap$ **X** = **X** for any named set **X**.

Proposition 3.11. (*Commutative Law*) **X** $\cap$ **Y** = **Y** $\cap$ **X** for any named sets **X** and **Y**.

Proposition 3.12 (*Associative Law*). **X** $\cap$ (**Y** $\cap$ **Z**) = (**X** $\cap$ **Y**) $\cap$ **Z** for any named sets **X**, **Y**, and **Z**.

Proposition 3.13 (*Monotone Law*). If **X** $\subseteq$ **Z**, then **Y** $\cap$ **X** $\subseteq$ **Y** $\cap$ **Z** for any named sets **X**, **Y**, and **Z**.

Corollary 3.5. If **X** $\subseteq$ **Z**, then **Z** $\cap$ **X** = **X** for any named sets **X** and **Z**.

Corollary 3.6. If **X** , **Y** $\subseteq$ **Z**, then **Y** $\cap$ **X** $\subseteq$ **Z** for any named sets **X**, **Y**, and **Z**, i.e., the intersection of named subsets of a named set **X** is a named subset of **X**.

However, in contrast to the union, the intersection of binormalized named sets is not always binormalized. That is why for proper modeling of events, we need to define binormalized intersection of named sets.

Definition 3.11. A named set **Y** = (Y, b, J) is called the *binormalized intersection* of named sets **X** = (X, a, I) and **Z** = (Z, c, K) if it is the union of all named sets that are unit named subsets of both **X** and **Z**.

It is denoted by **Y** = **X** $\cap_{\mathbf{B}}$ **Z**.

Proposition 3.14. The binormalized intersection of binormalized named sets is binormalized.

Proposition 3.15. The binormalized intersection of named subsets of a named set **X** = (X, r, I) is a named subset of a **X**.

Proposition 3.16. (*Idempotent Law*) **X** $\cap_{\mathbf{B}}$ **X** = **X** for any binormalized named set **X**.

Proposition 3.17. (*Annihilation Law*) **X** $\cap_{\mathbf{B}} \Lambda = \Lambda \cap_{\mathbf{B}}$ **X** $= \Lambda$ for any binormalized named set **X**.

Proposition 3.18. (*Commutative Law*) **X** $\cap_{\mathbf{B}}$ **Y** = **Y** $\cap_{\mathbf{B}}$ **X** for any binormalized named sets **X** and **Y**.

Proposition 3.19 (*Associative Law*). **X** $\cap_{\mathbf{B}}$ (**Y** $\cap_{\mathbf{B}}$ **Z**) = (**X** $\cap_{\mathbf{B}}$ **Y**) $\cap_{\mathbf{B}}$ **Z** for any binormalized named sets **X**, **Y**, and **Z**.

Proposition 3.20 (*Monotone Law*). If **X** $\subseteq_{\mathbf{B}}$ **Z**, then **Y** $\cap_{\mathbf{B}}$ **X** $\subseteq_{\mathbf{B}}$ **Y** $\cap_{\mathbf{B}}$ **Z** for any binormalized named sets **X**, **Y**, and **Z**.

Corollary 3.7. If **X** $\subseteq_B$ **Z**, then **Y** $\cap_B$ **X** $\subseteq_B$ **Z** for any binormalized named sets **X**, **Y**, and **Z**, i.e., if **X** is a binormalized named subset of **Z**, then **Y** $\cap_B$ **X** is a binormalized named subset of **Z** for any binormalized named sets **X**, **Y**, and **Z**.

Corollary 3.8. If **X** $\subseteq_B$ **Z**, then **Z** $\cap_B$ **X** = **X** for any binormalized named sets **X**, **Y**, and **Z**.

Proposition 3.21. X $\subseteq_B$ **Y** $\cup$ **X** and **X** $\cap_B$ **Y** $\subseteq_B$ **X** for any named sets **X** and **Y**.

Definition 3.12 [12]**.** A named set **Y** = (Y, q, J) is called the *difference* of named sets **X** = (X, r, I) and **Z** = (Z, p, K) if $Y = X \setminus Z$, $J = I \setminus K$, and $q = r \cap ((X \setminus Z) \times (I \setminus K))$.

It is denoted by **Y** = **X \ Z**.

Example 3.12. Subtraction of multisets in the sense of Hickman [39] gives an example of the difference of the named sets that represent these multisets (cf., for example, [34]) for a definition of subtraction of multisets).

Proposition 3.22. The difference of named subsets of a named set **X** = (X, r, I) is a named subset of a **X**.

Lemma 3.4. For any named set **X** = (X, r, I), we have **X \ X**= $\Lambda = (\varnothing, \varnothing, \varnothing)$.

However, in contrast to the union, the difference of binormalized named sets is not always binormalized. That is why for proper modeling of events, we need to define binormalized difference of named sets.

Definition 3.13. A named set **Y** = (Y, b, J) is called the *binormalized difference* of named sets $\mathbf{X} = (X, a, I)$ and $\mathbf{Z} = (Z, c, K)$ if it is the union of all named sets that are unit named subsets of **X** but not of **Z**.

It is denoted by **Y** = **X** $\setminus_B$ **Z.**

Lemma 3.5. For any binormalized named set **X** = (X, r, I), we have **X** $\setminus_B$ **X** = $\Lambda = (\varnothing, \varnothing, \varnothing)$.

Proposition 3.23. For any binormalized named sets **X** = (X, r, I) and **Y** = (Y, b, J), we have

a) (**X** $\setminus_B$ **Y**) $\cap_B$ **Y** = $\Lambda = (\varnothing, \varnothing, \varnothing)$

b) **X** = (**X** $\setminus_B$ **Y**) $\cup$ (**X** $\cap_B$ **Y**)

c) **X** $\cup$ **Y** = (**X** $\setminus_B$ **Y**) $\cup$ (**Y** $\setminus_B$ **X**) $\cup$ (**X** $\cap_B$ **Y**)

Corollary 3.9. For any binormalized named sets **X** = (X, r, I) and **Y** = (Y, b, J), we have

$$(\mathbf{X} \setminus_B \mathbf{Y}) \cap_B (\mathbf{Y} \setminus_B \mathbf{X}) = \Lambda$$

Corollary 3.10. For any binormalized named sets **X** = (X, r, I) and **Y** = (Y, b, J), we have

$$(\mathbf{X} \setminus_B \mathbf{Y}) \cap_B (\mathbf{X} \cap_B \mathbf{Y}) = \Lambda$$

Proposition 3.23. The binormalized difference of binormalized named sets is binormalized.

Proposition 3.24. The binormalized difference of binormalized named subsets of a binormalized named set **X** = (X, r, I) is a binormalized named subset of a **X**.

3.5 Similar to conventional probability, ample probability demands specific systems of named sets.

Definition 3.14. A system ***B*** of named sets is called a *named set algebra* if it is closed under intersection, union and difference of named sets, i.e. it satisfies the following conditions:

(NR1) **R**, **Q** $\in \mathbf{B}$ implies **R** $\cap$ **Q** $\in \mathbf{B}$.

(NR2) **R**, $D \in \mathbf{B}$ implies **R** $\cup$ **Q** $\in \mathbf{B}$.

(NR2) **R**, $D \in \mathbf{B}$ implies **R \ Q** $\in \mathbf{B}$.

Example 3.13. If **X** is a named set, then the set $2^{\mathbf{X}}$ of all named subsets of **X** is a named set algebra.

Lemma 3.6. For any set algebra $\boldsymbol{B}$, we have $\Lambda = (\varnothing, \varnothing, \varnothing) \in \boldsymbol{B}$.

Definition 3.15. A system $\boldsymbol{B}$ of binormalized named sets is called a *binormalized named set algebra* if it is closed under binormalized intersection, union and binormalized difference of named sets, i.e. it satisfies the following conditions:

(NR1) $\mathbf{R}, \mathbf{Q} \in \boldsymbol{B}$ implies $\mathbf{R} \cap_{\mathbf{B}} \mathbf{Q} \in \boldsymbol{B}$.

(NR2) $\mathbf{R}, \mathbf{Q} \in \boldsymbol{B}$ implies $\mathbf{R} \cup \mathbf{Q} \in \boldsymbol{B}$.

(NR3) $\mathbf{R}, \mathbf{Q} \in \boldsymbol{B}$ implies $\mathbf{R} \setminus_{\mathbf{B}} \mathbf{Q} \in \boldsymbol{B}$.

Lemma 3.6. For any binormalized named set algebra $\boldsymbol{B}$, we have $\Lambda = (\varnothing, \varnothing, \varnothing) \in \boldsymbol{B}$.

Example 3.14. If **X** is a binormalized named set, then the set $\{\mathbf{X}, \Lambda\}$ is a named set algebra.

Definition 3.16. A named set ring $\boldsymbol{B}$ with a unit element, i.e., an element **E** from $\boldsymbol{B}$ such that for any **R** from $\boldsymbol{B}$, we have $\mathbf{R} \cap \mathbf{E} = \mathbf{R}$, is called a *named set field*.

Example 3.15. If **X** is a named set, then the set $2^{\mathbf{X}}$ of all named subsets of **X** is a named set field with **X** as the unit element.

Definition 3.16. A binormalized named set ring $\boldsymbol{B}$ with a unit element, i.e., an element **E** from $\boldsymbol{B}$ such that for any **R** from $\boldsymbol{B}$, we have $\mathbf{R} \cap_{\mathbf{B}} \mathbf{E} = \mathbf{R}$, is called a *binormalized named set field*.

Example 3.16. If **X** is a binormalized named set, then the set $\{\mathbf{X}, \Lambda\}$ is a binormalized named set field.

For convenience, we consider only binormalized set algebras, elements of which are binormalized named subsets of some binormalized named set **U**. This name set can be finite or infinite. In this paper, we utilize just finite name sets. The ample probability theory with infinite name sets is studied elsewhere.

4. Axioms for Ample Probability

To build a mathematical model of events, we use named sets assuming that there is a space of all events under consideration. At first, we assume that there are a finite number of events.

Let us take a named set $\mathbf{U} = (U, f, V)$ and the set $2^{\mathbf{U}}$ of all named subsets of **U** [12].

Definition 4.1. An *event* in **U** is a binormalized named subset $\mathbf{K} = (K, r, T)$ of **U**.

The carrier K of the event (named set) **K** consists of tentative initial conditions while the reflector T of the event (named set) **K** consists of possible outcomes of this event.

It is natural to make the following assumptions:

(1) In an event, any initial conditions result in (are connected to) some outcomes.

(2) There are no outcomes without initial conditions.

(3) There are no outcomes without their connections to initial conditions.

It means that we do not consider events with empty initial conditions and/or empty outcomes and/or empty connections between initial conditions and outcomes. Thus, the named set model of an event can be only a binormalized named set.

Propositions 3.2, 3.11 and 3.20 imply the following result.

Proposition 4.1. If **U** is a binormalized named set, then the set $2^{\mathbf{BU}}$ of all binormalized named subsets of **U** is the maximal binormalized named set field in **U**.

Definition 4.2. An *elementary event* in **U** is a unit named subset **V** = (*V*, *v*, *H*) of **U**.

Properties of named sets imply the following result.

Proposition 4.2. Any event **Q** in **U** is the union of elementary events in **U**.

Indeed, by the definition of a binormalized named set, any element from the support of **Q** belongs to the support of some elementary event in **U** and any element from the set of names of **Q** belongs to the set of names of some elementary event in **U**. Finally, any connection between two elements from the naming correspondence of **Q** belongs to the naming correspondence of some elementary event in **U** because **U** and consequently, **Q** are set-theoretical named sets.

Let us assume a binormalized named set field ***F*** on **U** is chosen. By tradition, elements from ***F***, i.e., binormalized named subsets of **U** that belong to ***F***, are called *random events*.

Definition 4.3. a) A function from ***F*** into the set ***R*** of real numbers is called an *ample probability function*, if it satisfies the following axioms:

AP 1. (*Normalization*) P(**U**) = 1.

AP 2. (*Monotonicity*) For all events **K**, **Q** $\in$ ***F***, if **K** $\subseteq_{\mathbf{B}}$ **Q**, then P(**K**) $\leq$ P(**Q**)

AP 3. (*Finite additivity*)

P(**K** $\cup$ **Q**) = P(**K**) + P(**Q**)

for all events **K**, **Q** $\in$ ***F*** such that

K $\cap_{\mathbf{B}}$ **Q** = Λ.

b) The triad (**U**, ***F***, P) is called an *ample probability space*.

When the carrier U of a binormalized named set **U** = (U, f, V) consists of one element, then there is a one-to-one correspondence between binormalized named subsets of **U** and subsets of the set V, as well as a one-to-one correspondence between binormalized named fields in **U** and set fields in V. This correspondence allows proving the following result.

Theorem 4.1. If events are reduced to their outcomes, the ample probability function defines a unique probability function, which satisfies Kolmogorov's axioms P1-P3.

Proof. To prove the theorem, we remind Kolmogorov's axioms [43].

Let us consider a set Ω and a set field (or set algebra) **F** on Ω, i. e., a set of subsets of Ω that has Ω as a member, and that is closed under complementation (with respect to Ω) and union. Elements from **F** are called random events.

A function from a set field **F** to the set ***R*** of real numbers is called a *probability function*, if it satisfies the following axioms:

P 1. (*Non-negativity*) $P(A) \geq 0$, for all $A \in$ **F**.

P 2. (*Normalization*) $P(\Omega) = 1$.

P 3. (*Finite additivity*)

$P(A \cup B) = P(A) + P(B)$

for all sets A, $B \in$ **F** such that

$A \cap B = \emptyset$.

Now let us take a named set **U** = (U, f, V) where $V = \Omega$, $U = \{c\}$ and $f = \{(c, e);\ e \in \Omega\}$. There is a one-to-one correspondence q between subsets of Ω and binormalized named subsets of **U**. Indeed, if $X \subseteq \Omega$, then $q(X)$ = **X** $\subseteq_{\mathbf{B}}$ **U** where **X** = (U, g, W) where $V = X$, $U = \{c\}$ and $f = \{(c, e);\ e \in X\}$. We also put $q(\emptyset) = \Lambda$.

Properties of named sets allow proving the following properties:

$$q(X \cup Y) = q(X) \cup q(Y)$$

$$q(X \cap Y) = q(X) \cap_{\mathbf{B}} q(Y)$$

$$q(X \setminus Y) = q(X) \setminus_{\mathbf{B}} q(Y)$$

Taking a subset $\boldsymbol{L}$ of the set 2^{Ω} of all subsets of Ω, we assign to it the set $q(\boldsymbol{L})$ of binormalized named subsets of **U** by the following formula

$$q(\boldsymbol{L}) = \{\, q(X);\ X \in \boldsymbol{L}\} = \{\mathbf{K};\ \mathrm{S}(\mathbf{K}) = U \ \&\ \mathrm{N}(\mathbf{K}) \in \boldsymbol{L}\}$$

If $\boldsymbol{L}$ is a set algebra, then $q(\boldsymbol{L})$ is a binormalized named set algebra, i.e., it satisfies conditions NR1 – NR3. Indeed, we have:

(NR1) if $q(X)$, $q(Y) \in q(\boldsymbol{L})$, then $q(X) \cap_{\mathbf{B}} q(Y) \in q(\boldsymbol{L})$ because $q(X) \cap_{\mathbf{B}} q(Y) = q(X \cap Y)$ and $X \cap Y \in \boldsymbol{L}$.

(NR2) if $q(X)$, $q(Y) \in q(\boldsymbol{L})$, then $q(X) \cup q(Y) \in q(\boldsymbol{L})$ because $q(X) \cup q(Y) = q(X \cup Y)$ and $X \cup Y \in \boldsymbol{L}$.

(NR3) if $q(X)$, $q(Y) \in q(\boldsymbol{L})$, then $q(X) \setminus_{\mathbf{B}} q(Y) \in q(\boldsymbol{L})$ because $q(X) \setminus_{\mathbf{B}} q(Y) = q(X \setminus Y)$ and $X \setminus Y \in \boldsymbol{L}$.

Moreover, if $\boldsymbol{L}$ is a set field, then $q(\boldsymbol{L})$ is a binormalized named set field. Indeed, if Ω belongs to $\boldsymbol{L}$, then **U** also belongs to $q(\boldsymbol{L})$.

Let us assume that an ample probability function P: $q(\boldsymbol{L}) \to \boldsymbol{R}$ is defined. This function induces the function p: $\boldsymbol{L} \to \boldsymbol{R}$ by the following rule:

$p(X) = P(q(X))$ for any $X \in \boldsymbol{L}$

Let us show that p is a probability function, which satisfies Kolmogorov's axioms P1 – P3.

Axioms P2: $p(\Omega) = 1$ because $q(\Omega)$ = **U** and by Axiom AP1, $P(\mathbf{U}) = 1$.

Axioms P1: At first we show that $p(\emptyset) = 0$. Indeed, $q(\emptyset) = \Lambda$. At the same time, $\mathbf{U} \cup \Lambda = \mathbf{U}$ and $\mathbf{U} \cap_{\mathbf{B}} \Lambda = \Lambda$. Thus, by Axiom AP3, $P(\mathbf{U}) = P(\mathbf{U}) + P(\Lambda)$. Consequently, $P(\Lambda) = 0$ because by Axiom AP1, $P(\mathbf{U}) = 1$. Consequently, $p(\emptyset) = p(q(\emptyset)) = P(\Lambda) = 0$.

Axioms P3: Let us take arbitrary sets A, $B \in \boldsymbol{L}$ such that $A \cap B = \emptyset$. Then Axiom AP3 implies the following equalities

$p(A \cup B) = p(q(A \cup B)) = p(q(A) \cup q(B)) = p(q(A)) + p(q(B)) = p(A) + p(B)$

because $A \cap B = \emptyset$ implies $q(A) \cap_{\mathbf{B}} q(B) = \Lambda$

Theorem is proved.

Note that in the setting of Theorem 4.1, axiom AP2 follows from axiom AP3.

Informally, the condition that the carrier U of a binormalized named set **U** = (U, f, V) consists of one element means that all events have the same initial conditions. This is actually the presupposition of the classical probability as in their classical works, von Mises and Kolmogorov considered a complex of initial conditions treating events as what happens or what can happen when the given initial conditions are realized [43, 63]. As a result, initial conditions are supposed to be the same for all considered events.

As it is proved that Kolmogorov's axioms P1 – P3 are consistent and independent [43], we have the following result.

Theorem 4.2. Axioms AP1 – AP4 of ample probability are consistent and independent.

Proof. Consistency. By Theorem 4.1, axioms AP1 – AP4 for the maximal binormalized named set field in a binormalized named set **U** = (U, f, E) coincide with Kolmogorov's axioms P1 – P3 when the carrier U of **U** consists of one element. As it proved that axioms P1 – P3 are consistent [43] axioms AP1 – AP4 are also consistent because they have a model.

Independence. Axiom AP1: Taking the unit binormalized named set **U** = (U = {0}, f = {(0, 1)}, E = {1}) and defining P(**U**) = 2 and $P(\Lambda) = 0$, we see that axioms AP1, AP3 and AP4 are true while the axiom AP2 is not.

Axiom AP2: Let us consider the binormalized named set **U** = (U = {0}, f = {(0, 1), (0, 2)}, E = {1, 2}) and define the function P: $2^{\mathbf{BU}} \rightarrow \boldsymbol{R}$ in the following way:

$$P(\mathbf{U}) = 1$$

$$P(\Lambda) = 0$$

$$P(\mathbf{V}) = 2$$

$$P(\mathbf{W}) = 2$$

where **V** = ({0}, {(0, 1)}, {1}) and **W** = ({0}, {(0, 2)}, {2}).

For this function, axioms AP1 and AP2 are true by definition. Axiom AP4 is true because there are no named subsets **K** and **Q** of **U** for which **K** $\cap_{\mathbf{B}}$ **Q** = Λ. However, axiom AP3 is not true because P(**U**) = 1, P(**V**) = 2 while **K** $\subseteq$ **Q**.

Axiom AP3: Let us consider the binormalized named set **U** = (U = {0, 3}, f = {(0, 1), (3, 2)}, E = {1, 2}) and define the function P: $2^{\mathbf{BU}} \rightarrow \boldsymbol{R}$ in the following way:

$$P(\mathbf{U}) = 1$$

$$P(\Lambda) = 0$$

$$P(\mathbf{V}) = 1$$

$$P(\mathbf{W}) = 1$$

Where **V** = ({0}, {(0, 1)}, {1}) and **W** = ({3}, {(3, 2)}, {2}).

For this function, axioms AP1 and AP2 are true by definition. Axiom AP3 is true because P(**U**) = P(**V**) = P(**W**) while axiom AP3 is not true because **K** $\cap_{\mathbf{B}}$ **Q** = Λ but P(**U**) $\neq$ P(**V**) + P(**W**).

Theorem is proved.

Note that while for conventional probability monotonicity follows from finite additivity, for ample probability, this is not true.

5. Properties of Ample Probability

Let us explore ample probability.

Proposition 5.1. $P(\Lambda) = 0$.

Indeed, $\mathbf{U} \cup \Lambda = \mathbf{U}$ and $\mathbf{U} \cap_{\mathbf{B}} \Lambda = \Lambda$. Thus, by Axiom AP4, $P(\mathbf{U}) = P(\mathbf{U}) + P(\Lambda)$. Consequently, $P(\Lambda) = 0$ because by Axiom AP1, $P(\mathbf{U}) = 1$.

Some properties of ample probability are similar to properties of conventional probability.

Proposition 5.2. For any event $\mathbf{K} \in \boldsymbol{F}$, we have $0 \leq P(\mathbf{K}) \leq 1$, i.e., P is a function from $\boldsymbol{F}$ into the interval [0, 1].

Proof. If $\mathbf{K} \in \boldsymbol{F}$, then $\Lambda \subseteq_{\mathbf{B}} \mathbf{K} \subseteq_{\mathbf{B}} \mathbf{U}$. Thus, by Axiom AP2, $P(\Lambda) \leq P(\mathbf{K}) \leq P(\mathbf{U})$. By Axiom AP1, $P(\mathbf{U}) = 1$, while by Proposition 5.1, $P(\Lambda) = 0$. Consequently, $0 \leq P(\mathbf{K}) \leq 1$.

Proposition is proved.

Corollary 5.1. (*Non-negativity*) $P(\mathbf{Q}) \geq 0$, for all $\mathbf{Q} \in \boldsymbol{F}$.

Proposition 5.3. For any events $\mathbf{K}, \mathbf{Q} \in \boldsymbol{F}$, we have:

$$P(\mathbf{K} \cap_{\mathbf{B}} \mathbf{Q}) \leq P(\mathbf{K}) \leq P(\mathbf{K} \cup \mathbf{Q})$$

Indeed, as by Proposition 3.21, $\mathbf{K} \cap_{\mathbf{B}} \mathbf{Q} \subseteq_{\mathbf{B}} \mathbf{K} \subseteq_{\mathbf{B}} \mathbf{K} \cup \mathbf{Q}$, by Axiom AP2, we obtain $P(\mathbf{K} \cap_{\mathbf{B}} \mathbf{Q}) \leq P(\mathbf{K}) \leq P(\mathbf{K} \cup \mathbf{Q})$.

Proposition 5.4. For any event $\mathbf{K} \in \boldsymbol{F}$, we have $P(\mathbf{K}) = 1 - P(C\mathbf{K})$ where $C\mathbf{K} = \mathbf{U} \setminus_{\mathbf{B}} \mathbf{K}$.

Proof. If $\mathbf{K} \in \boldsymbol{F}$, then $\mathbf{K} \subseteq \mathbf{U}$, $\mathbf{K} \cap_{\mathbf{B}} C\mathbf{K} = \Lambda$ and $\mathbf{K} \cup C\mathbf{K} = \mathbf{U}$. Thus, by Axiom AP3, $P(\mathbf{K}) + P(C\mathbf{K}) = P(\mathbf{U})$, while by Axiom AP1, $P(\mathbf{U}) = 1$. Consequently, $P(\mathbf{K}) = 1 - P(C\mathbf{K})$.

Proposition is proved.

Proposition 5.5. For any events $\mathbf{K}, \mathbf{Q} \in \boldsymbol{F}$, we have:

$$P(\mathbf{K} \cup \mathbf{Q}) = P(\mathbf{K}) + P(\mathbf{Q}) - P(\mathbf{K} \cap_{\mathbf{B}} \mathbf{Q})$$

Proof. By Proposition 3.19,

$$\mathbf{X} \cup \mathbf{Y} = (\mathbf{X} \setminus_{\mathrm{B}} \mathbf{Y}) \cup (\mathbf{Y} \setminus_{\mathrm{B}} \mathbf{X}) \cup (\mathbf{X} \cap_{\mathbf{B}} \mathbf{Y})$$

As by Corollary 3.9 and 3.10 the following equalities are true

$$(\mathbf{X} \setminus_{\mathbf{B}} \mathbf{Y}) \cap_{\mathbf{B}} (\mathbf{Y} \setminus_{\mathbf{B}} \mathbf{X}) = \Lambda$$

$$(\mathbf{Y} \setminus_{\mathbf{B}} \mathbf{X}) \cap_{\mathbf{B}} (\mathbf{X} \cap_{\mathbf{B}} \mathbf{Y}) = \Lambda$$

$$(\mathbf{X} \setminus_{\mathbf{B}} \mathbf{Y}) \cap_{\mathbf{B}} (\mathbf{X} \cap_{\mathbf{B}} \mathbf{Y}) = \Lambda$$

by Axiom AP3, we have

$$P(\mathbf{X} \cup \mathbf{Y}) = P(\mathbf{X} \setminus_{\mathbf{B}} \mathbf{Y}) + P(\mathbf{Y} \setminus_{\mathbf{B}} \mathbf{X}) + P(\mathbf{X} \cap_{\mathbf{B}} \mathbf{Y})$$

$$P(\mathbf{X}) = P(\mathbf{X} \setminus_{\mathbf{B}} \mathbf{Y}) + P(\mathbf{X} \cap_{\mathbf{B}} \mathbf{Y}) \qquad (5.1)$$

$$P(\mathbf{Y}) = P(\mathbf{Y} \setminus_{\mathbf{B}} \mathbf{X}) + P(\mathbf{X} \cap_{\mathbf{B}} \mathbf{Y}) \qquad (5.2)$$

Adding (5.1) and (5.2), we obtain

$$P(\mathbf{X}) + P(\mathbf{Y}) = P(\mathbf{X} \setminus_{\mathbf{B}} \mathbf{Y}) + P(\mathbf{Y} \setminus_{\mathbf{B}} \mathbf{X}) + 2P(\mathbf{X} \cap_{\mathbf{B}} \mathbf{Y})$$

Consequently,

$$P(\mathbf{K} \cup \mathbf{Q}) = P(\mathbf{K}) + P(\mathbf{Q}) - P(\mathbf{K} \cap_{\mathbf{B}} \mathbf{Q})$$

Proposition is proved.

Proposition 5.6. For any events $\mathbf{K}, \mathbf{Q} \in \boldsymbol{F}$, we have:

$$P(\mathbf{K} \cup \mathbf{Q}) = P(\mathbf{K}) + P(\mathbf{Q} \setminus_{\mathbf{B}} \mathbf{K})$$

Proof. By properties of named sets, **K** ∪ **Q** = **K** ∪ (**Q** $\backslash_B$ **K**) and **K** $\cap_B$ (**Q** $\backslash_B$ **K**) = Λ. Thus, by Axiom AP3, P(**K** ∪ **Q**) = P(**K** ∪ (**Q** $\backslash_B$ **K**)) = P(**K**) + P(**Q** $\backslash_B$ **K**)

Proposition is proved.

Corollary 5.2. For any events **K**, **Q** ∈ F, if **K** ⊆ **Q**, then

$$P(\mathbf{Q}) = P(\mathbf{K}) + P(\mathbf{Q} \backslash_B \mathbf{K})$$

Corollary 5.3. For any events **K**, **Q** ∈ F, if **K** ⊆ **Q** and P(**K**) = P(**Q**), then

$$\mathbf{Q} \backslash_B \mathbf{K} = \Lambda$$

6. Case Studies

There are many problems that demand the calculation of probability while the event analysis and structuration is the chief and most demanding task. Current theories overlook the structuration argument and specialists fuse (and sometimes confuse) the calculations of event structures with the calculus of probability. Here we show how structuration allows accurate calculation of probabilities.

Three machines *A, B* and *C* respectively output 45%, 35% and 20% of the total production in a factory. The percentages of faulty items are respectively 2%, 3% and 1% in each of them. One item is selected randomly, which is the considered event, and we must calculate the probability that it is defective.

To properly do this, we have to go beyond the outcome "defective output" and explicate the event structure taking into account initial conditions. Inclusion of initial conditions brings us to ample probability.

At the beginning, we have an event of the form

$$S = (E_i\,, R, E_o)$$

Where E_i means "input materials for machines A, B and C", E_o means "produced items", and R denotes the process of production.

Event S is the union of the elementary events of the form

$$S_e = (E_i\,, R, E_{eo})$$

Where E_i means "input materials for machines A, B and C", E_{eo} means "a produced item", and R denotes the process of production.

On the first level, the event structure (cf. Formula 6.1) reflects the decomposition of the event S into the events S_P and S_F of producing good quality and defective items, respectively. This is reflected by splitting the set of names into two parts.

$$S = S_G \cup S_F \tag{6.1}$$

As a result, we have two events – production of a good quality items

$$S_G = (E_i\,, R, E_{Go})$$

and production of a defective items

$$S_F = (E_i\,, R, E_{Fo})$$

Where E_i are the same as before, E_{Go} means "produced good quality items", and E_{Fo} means "produced defective items".

Event S_G is the union of the elementary events either of the form

$$S_{Ge} = (E_i , R, E_{eGo})$$

or of the form

$$S_{Fe} = (E_i , R, E_{eFo})$$

In the same way, we decompose the event S into the events S_A , S_B and S_C of production by machines A, B and C, respectively. This is reflected by splitting the set of names into three parts.

$$S = S_A \cup S_B \cup S_C \tag{6.2}$$

As a result, we have three events – production of a good quality items

$$S_X = (E_{X_i} , R_X, E_{X_o})$$

Where E_{Xi} means "input materials for the machine X", E_{Xo} means "produced items by the machine X", and R_X denotes the process of production by the machine X (here $X = A, B, C$).

On the next level, we blow up the event S_G demonstrating that the perfect item is produced by one of the machines A, B and C. It gives us a decomposition of the event S_G into three possible events S_{AG}, S_{BG} and S_{CG} producing good quality items.

$$S_G = S_{AG} \cup S_{BG} \cup S_{CG} \tag{6.3}$$

Here for $X = A, B, C$, we have

$$S_{XG} = (E_{XG_i} , R_X , E_{XG_o})$$

Where E_{X_i} means "input materials for the machine X", E_{X_o} means "produced good quality items by the machine X", and R_X denotes the process of production by the machine X ($X = A, B, C$).

Production of good quality items by the machine X means production of items by the machine X ($X = A, B, C$), and production of good quality items. This results in the intersections of events S_G and S_X for $X = A, B, C$.

$$S_{XG} = S_G \cap S_X$$

The same operation is applied to the event R_F .

$$S_F = S_{AF} \cup S_{BF} \cup S_{CF} \tag{6.4}$$

Production of defective items by the machine X means production of items by the machine X ($X = A, B, C$), and production of good quality items. This results in the intersections of events S_F and S_X for $X = A, B, C$.

$$S_{XF} = S_F \cap S_X$$

It gives us Formula 6.5.

$$S = S_P \cup S_F = [(S_a \cap S_{Pa}) \cup (S_b \cap S_{Pb}) \cup (S_c \cap S_{Pc})] \cup [(S_a \cap S_{Fa}) \cup (S_b \cap S_{Fb}) \cup (S_c \cap S_{Fc})] \tag{6.5}$$

As the problem requires only the probability $P(R_F)$ of the defective production, we need only the second half of the whole event S:

$$S_F = (S_a \cap S_{Fa}) \cup (S_b \cap S_{Fb}) \cup (S_c \cap S_{Fc}) \tag{6.6}$$

Assuming the classical (combinatorial) approach to probability (cf., for example, [14, 36], initial conditions give us the following probabilities: $P(S_a) = 0.45$, $P(S_{Fa}) = 0.02$, $(P(S_b) = 0.35$, $P(S_{Fb}) = 0.03$, $(P(S_c) = 0.20$, $P(S_{Fc}) = 0.01$. This allows computation of the necessary probability using Formula (6.7).

$$P(S_F) = (P(S_a)\cdot P(S_{Fa})) + (P(S_b)\cdot P(S_{Fb})) + (P(S_c)\cdot P(S_{Fc})) =$$
$$(0.45 \times 0.02) + (0.35 \times 0.03) + (0.20 \times 0.01) = 0.009 + 0.0105 + 0.002 = 0.0215 \qquad (6.7)$$

Calculating the event in advance of the relative probability, such as in the considered example, constitutes a significant methodological progress due to the strategy when the structural and probabilistic questions are tackled in two different stages.

Note that this example shows that the classical combinatorial approach to probability implicitly includes initial conditions in the description of events but assuming constancy of these conditions represents events only by their outcomes.

7. Conclusion

Employing dynamic or causal structuring of events and their representation in a form of binormalized named sets, we constructed a probability function called ample probability for such events and developed elements of an axiomatic ample probability theory. We showed that axioms that characterize ample probability are consistent and independent. It is also proved that in the limit, i.e., when all named sets representing events are singlenamed sets, the axiom system for ample probability becomes Kolmogorov's axiom system for conventional probability.

Although Kolmogorov's axioms are the most popular in probability theory, there are other systems of axioms suggested for probability [14]. That is why it is natural to suggest that the system of axioms constructed in this paper describes the standard ample probability theory. This brings us to the interesting problem of developing other axioms for ample probability.

8. Conflicts of Interest

The author(s) report(s) no conflicts of interest(s) and the author along are responsible for the content and writing of the paper.

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