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Symmetric Hyperprobabilities

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Abstract

The aim of this work is further advancement of hyperprobability theory by including negative hyperprobabilities in the scope of this theory and building an axiomatic theory of symmetric hyperprobabilities. Hyperprobabilities are more general than conventional probabilities being constructed using real hypernumbers instead of real numbers. This extension allows essential extension of the scope of probability theory and its applications. Another extension of probability theory is related to negative probabilities. Although classical probability is a function with values in the interval [0, 1], researchers found that negative probabilities could be a useful tool in physics, economics and some other areas. In this paper, we synthesize both approaches developing theory of symmetric hyperprobabilities, which can take both positive and negative values in hypernumbers. In Section 2, some constructions and concepts from the theory of hypernumbers and extrafunctions are presented. Section 3 develops foundations for symmetric hyperprobabilities. In Section 4, symmetric hyperprobabilities are introduced and studied.

Key words: Negative probability, Hyperprobability, Annihilation, Symmetry, Axiom.

1. Introduction

Although probability theory is currently an advanced mathematical discipline, which provides foundation for statistics being an important instrument for natural and social sciences, as well as for business, industry and engineering, there is no unified conception of the term *probability* and there is no unified mathematical system that formalizes the concept *probability*. To obtain a more complete and comprehensive theory, hyperprobabilities have been constructed, axiomatized and explored [1-3]. While probabilities take values in non-negative real numbers, conventional hyperprobabilities take values in non-negative real hypernumbers.

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Real and complex hypernumbers form a far reaching extension of real and complex numbers allowing differentiation and integration of a much wider scope of functions in comparison with real and complex numbers. As a result, it becomes possible to solve much more differential and integral equations and rigorously work with infinity, which is very important for contemporary physics [4-5].

In turn, the conception of hyperprobability provides a solution for a variety of problems that researchers in diverse disciplines have with the concept of probability. It also essentially extends the capabilities of probability theory giving an efficient tool for investigation of all sequences of events or experiments and not only of random events. Additionally, hyperprobability forms foundations for the frequency approach to probability. Conventionally, probability is treated as a function with values in the interval [0, 1]. However, researchers found that probabilities values of which go beyond this interval could be useful in variety of situations. The most important are probabilities, which have been efficiently utilized in a quantity of theoretical and practical areas although they take negative values. These probabilities are often called negative or extended. The first who unconsciously encountered probabilities that take negative values in quantum mechanics was Hermann Weyl [6]. The next encounter with negative probabilities happened in works of Paul Dirac [7] and Werner Heisenberg [8], who also missed their significance and possibility to take negative values. In comparison with them, Eugene Wigner [9] came to the conclusion that quantum corrections often lead to negative probabilities.

However, the importance of Wigner's function for quantum physics was not recognized for a long time. However, time passed physicists more and more saw efficacy of negative probabilities in physics and little by little negative probabilities have become very popular among physicists finding application to a diversity of physical problems, including quantum field theory, quantum optics, statistical mechanics, nuclear theory and hydrodynamics (cf., for example, [10-13]). Dirac [14] and Feynman [15] described negative numbers as analogies for negative probabilities to demonstrate their meaningfulness in physics. However, the first mathematically consistent physical interpretations for probabilities that take negative values appeared much later [16]. There are also other physical interpretations for such probabilities [17-18].

Although some physicists prefer to use the term *quasi-probability* instead of calling such more general probabilities by the name negative or extended probabilities, probabilities that take negative values have been extensively used becoming more and more popular in physics (cf., for example, [15, 19-28]). Now negative probabilities are a commonplace tool for numerous physicists (cf., for example, [29-38]). Negative probabilities also appear as an important feature of quantum computation [33, 39-42] and system theory [43-45]. In addition, probabilities that take negative values, i.e., negative probabilities, came to social and behavioral sciences (cf., for example, [46, 47]), economics and finance [48-53].

At the same time, the concept of probability was extended in another direction. The main idea behind this concept is defining probability only for random events. The formal representation of randomness in the most popular axiomatics developed by Kolmogorov for probability treats random events as elements of the set algebra **F**, which consists of subsets of the set Ω of all elementary events and is defined by its formal properties [54]. However, this approach as well as other descriptions of random events, encountered many problems and limitations. To overcome these limitations and solve corresponding problems, the theory of hyperprobability was created fitting the concept a random event into a more general schema [1, 3, 55].

The main goal of this paper is to extend hyperprobability theory by including negative hyperprobabilities in the scope of this theory and building an axiomatic theory of symmetric hyperprobabilities, which include negative hyperprobabilities and negative probabilities. It is a natural and in some sense, needed step because investigation of relations between probability theory and science demonstrates needs in extending probability theory and the concept of probability [56-58].

This paper is organized as follows. In Section 2, some constructions and concepts from the theory of hypernumbers and extrafunctions are presented. Section 3 develops foundations for symmetric hyperprobabilities. In Section 4, symmetric hyperprobabilities are introduced and studied providing the first introduction to symmetric hyperprobability theory.

2. Real Hypernumbers as a Natural Extension of Real Numbers

Let us take the set $\boldsymbol{R}$ of all real numbers and consider the set $\boldsymbol{R}^{\omega} = \{(a_i)_{i\in\omega} ; a_i \in \boldsymbol{R}\}$ of all sequences of real numbers where $\omega = \{1, 2, 3, 4, 5,\ldots\}$ is the sequence of all natural (counting) numbers.

Definition 2.1. If sequences $\boldsymbol{a} = (a_i)_{i\in\omega}$ and $\boldsymbol{b} = (b_i)_{i\in\omega}$ belong to $\boldsymbol{R}^{\omega}$, then

$$\boldsymbol{a} \sim \boldsymbol{b} \leftrightarrow \lim_{i\to\infty} |a_i - b_i| = 0 .$$

The relation ~ is an equivalence relation used for defining real hypernumbers [4-5].

Definition 2.2. A *real hypernumber* is an equivalence class in the set $\boldsymbol{R}^{\omega}$ under the equivalence relation ~.

The set of all such equivalent classes is denoted by $\boldsymbol{R}_{\omega}$. An equivalence class may be determined and represented by its arbitrary member, that is, by an element from this equivalence class. Thus, real hypernumbers are treated as sets of equivalent sequences of real numbers in the same way as real numbers are treated as sets of equivalent fundamental sequences of rational numbers or rational numbers are treated as sets of equivalent fractions.

If $\boldsymbol{a} = (a_i)_{i\in\omega}$ is sequence of real numbers, then $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ is the hypernumber represented, determined and generated by this sequence. The sequence $\boldsymbol{a}$ is called a *representation* of the hypernumber α.

It is possible to introduce operations and relations in the set $\boldsymbol{R}_{\omega}$ of all real hypernumbers, which are similar to operations and relations in the set $\boldsymbol{R}$ of all real numbers.

Theorem 2.1. [4] The set $\boldsymbol{R}_{\omega}$ of all real hypernumbers is a partially ordered infinite dimensional vector space over the field $\boldsymbol{R}$ of real numbers.

It means that it is possible to add and subtract real hypernumbers, multiply them by real numbers, and compare some pairs of real hypernumbers. Note that in contrast to real numbers, not all real hypernumbers are comparable. The partial order in $\boldsymbol{R}_{\omega}$ is defined as follows:

If $\alpha, \beta \in \boldsymbol{R}_{\omega}$, then

$\alpha \leq \beta$ if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$, $\beta = \mathrm{Hn}(b_i)_{i\in\omega}$

and there is n such that

$a_i \leq a_i$ for all $i > n$.

Operations in $\boldsymbol{R}_{\omega}$ are defined as follows:

If $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ and $\beta = \mathrm{Hn}(b_i)_{i\in\omega}$, then

$\alpha + \beta = \mathrm{Hn}(a_i + b_i)_{i\in\omega}$

and

$\alpha - \beta = \mathrm{Hn}(a_i - b_i)_{i\in\omega}$.

If $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ and $r \in \boldsymbol{R}$, then

$r\alpha = \mathrm{Hn}(ra_i)_{i\in\omega}$.

It is proved that addition of real hypernumbers is commutative and associative, while multiplication of real hypernumbers by real numbers is distributive with respect to addition and subtraction [4].

3. Events and Antievents as the Base for Symmetric Hyperprobabilities

Let us consider a set Ω, which consists of two irreducible disjoint parts (subsets) Ω^+ and Ω^-, i.e., neither of these parts is equal to its proper subset. Elements from the set Ω^+ are called *elementary positive events*, while elements from the set Ω^- are called *elementary negative events* or *antievents*. Subsets of the set Ω^+ are called *positive events*, while subsets of the set Ω^- are called *negative events* or *antievents*. All other subsets of the set Ω are *mixed events*.

There are different examples of negative events (antievents). They are usually connected to negative objects. For instance, encountering or producing a negative object is a negative event.

An example of negative objects is given by antiparticles, which form the antimatter counterpart of quantum particles [59]. Thus, the electron e^- has the antiparticle, which is called the positron and denoted by e^+. The discovery of the positron has an interesting history. When Paul Dirac extended quantum mechanics to include special relativity, he derived a formula known as the Dirac equation. This equation had two solutions. One of them described the electron, while the other one predicted that an electron should have a positively charged counterpart [60]. However, other physicists did not accept this idea, criticized Dirac and even mocked at him for his innovation. It was not easy to find positrons because the positrons produced in natural radioactive decay quickly annihilate themselves with electrons, producing pairs of gamma rays. However, in spite of the negative attitude of the majority of physicists, this particle, the positron, was soon discovered in the cosmic radiation by Carl Anderson in 1932 [61].

Operations with money give another example of positive and negative events. Indeed, it is possible to consider receiving $100 as a positive event and giving $100 as a negative event. If somebody gives $100 and then receives $100, these events annihilate one another and the person has the same amount of money as before.

The area of software development gives one more example of negative objects. Namely, they are antipatterns, which are negative design patterns in software industry [62-63]. A software design pattern is an *antipattern* if it is a repeated pattern of action, process or structure that initially appears to be beneficial, but ultimately produces more bad consequences than beneficial results, and for which an alternative solution exists that is clearly documented, proven in actual practice and repeatable.

Observations of subatomic particles have persuasively demonstrated that when a particle meets its antiparticle, they annihilate each other and disappear, their combined rest energies becoming available to appear in other forms [59]. These processes are described by creation and annihilation operators.

Note that annihilation occurs not only in physics where particles and antiparticles annihilate one another, but also in ordinary life of people. For instance, Andrew has stocks of two companies. If in 2011, the first set of stocks gave profit \$10,000, while the second set of stocks dropped by \$10,000, then the combined income was \$0. The loss annihilated the profit.

Here is even a simpler example. Bonnie finds \$50 and loses \$20. As the result, the amount of her money has increased by \$30. The loss annihilated a part of the gain.

Thus, it is natural to assume that when an event contains an elementary event w and its antievent $-w$, then w and $-w$ annihilate one another. To formalize this situation, we introduce additional equality relation $\equiv$, which is determined by the following formula

$$A = \{v_i , w, -w; v_i, w \in \Omega \ \& \ i \in I\} \equiv B = \{v_i; v_i \in \Omega \ \& \ i \in I\}$$

Informally, the relation $A \equiv B$ means that if a set A contains an event w and its antievent $-w$, then w and $-w$ annihilate one another and the new set B is, in some sense, equal to the previous set A, i.e., sets A and B are equivalent with respect to annihilation of events and antievents.

For any set $X \subseteq \Omega$, we define:

$$X^+ = X \cap \Omega^+,$$

$$X^{--} = X \cap \Omega^{--},$$

$$-X = \{-w; w \in X\}$$

As it was defined above, if X is a positive event, then $-X$ is a negative event or antievent.

There are also *relative antievents*. Namely, $-X$ is an antievent for an event $X \subseteq \Omega$.

The phenomenon of annihilation of elementary events gives birth to the *union with annihilation* of two subsets X and Y of Ω by the following formula:

$$X \oplus Y = (X \cup Y) \setminus [(X \cap -Y) \cup (-X \cap Y) \cup (X \cap -X) \cup (-Y \cap Y)]$$

The *union with weak annihilation* of two subsets X and Y of Ω is defined by the following formula:

$$X \boxplus Y = (X \cup Y) \setminus [(X \cap -Y) \cup (-X \cap Y)]$$

Here the set-theoretical operation \ represents annihilation, while sets $X \cap -X$, $-Y \cap Y$, $X \cap -Y$ and $X \cap -Y$ are formed from annihilating entities, i.e., events and antievents. For instance, we have the following equalities:

$$X \oplus -X = X \boxplus -X = \emptyset,$$

$$\{w, -w\} \oplus \emptyset = \emptyset$$

while $\{w, -w\} \boxplus \emptyset = \{w, -w\}$.

Note that in the weak annihilation, an event annihilates its antievent only if they belong to different sets.

To define a probability or hyperprobability function, a subset **F** of the set 2^Ω of all subsets of Ω is selected. Elements from **F**, i.e., subsets of Ω that belong to **F** are called *random events* or in a general case, *admissible events* because hyperprobability theory allows assigning probabilities not only to random events [1-3].

In the axiomatic probability theory, there is no a commonly accepted approach to the definition of a random event, and the set **F** is formally described by axioms.

Elements from the set $\mathbf{F}^+ = \{X \in \mathbf{F}; X \subseteq \Omega^+\}$ are called *positive random* (*admissible*) *events*.

Elements from the set $\mathbf{F}^- = \{-A;\ A \in \mathbf{F}^+\}$ are called *negative random* (*admissible*) *events* or *random* (*admissible*) *antievents*.

Elements from Ω^+ that belong to the set $\mathbf{F}^+$ are called *elementary positive random* (*admissible*) *events* or *elementary random* (*admissible*) *events*.

Elements from Ω^- that belong to the set $\mathbf{F}^-$ are called *elementary negative random* (*admissible*) *events* or *elementary random* (*admissible*) *antievents*.

Note that the model is symmetric because positive events are, in this sense, antievents of the corresponding negative events, e.g., E is the antievent of $-E$.

If $A \in \mathbf{F}^+$, then $-A$ is called the *antievent* of A. In particular, if $w \in \Omega^+$, then $-w$ is the *antievent* of the event w.

Let us consider some properties of the introduced constructions.

Union with annihilation $\oplus$, union with weak annihilation $\boxplus$ and conventional union $\cup$ have similar properties although in a general case, these operations do not coincide.

Proposition 3.1. a) $X \oplus X \equiv X \boxplus X \equiv X$ for any subset X of Ω.

b) $X \oplus Y \equiv X \boxplus Y \equiv X \cup Y$ for any subsets X and Y of Ω.

c) $X \oplus Y \subseteq X \boxplus Y \subseteq X \cup Y$ for any subsets X and Y of Ω.

d) $X \boxplus \varnothing = X$ for any subset X of Ω.

e) $X \oplus \Omega \equiv X$ for any subset X of Ω.

f) $X \oplus \Omega = X \oplus \varnothing$ for any subset X of Ω.

g) $X \oplus (Y \oplus Z) \equiv (X \oplus Y) \oplus Z$ and $X \boxplus (Y \boxplus Z) \equiv (X \boxplus Y) \boxplus Z$ for any subsets X, Y and Z of Ω.

h) $X \oplus Y = Y \oplus X$ and $X \boxplus Y = Y \boxplus X$ for any subsets X and Y of Ω.

i) $X \oplus Y = X \boxplus Y$ if and only if both subsets X and Y of Ω do not have an event and its antievent.

At the same time, other properties of operations $\oplus$, $\boxplus$ and $\cup$ are different. For instance, operations $\cap$ and $\cup$ are not distributive with respect to the operations $\oplus$ and $\boxplus$.

Proposition 3.2. a) It is possible that $X \oplus X \neq X$ and $X \boxplus X \neq X$.

b) It is possible that $Z \cup (X \oplus Y) \neq (Z \cup X) \oplus (Z \cup Y)$ and $Z \cup (X \boxplus Y) \neq (Z \cup X) \boxplus (Z \cup Y)$ for some subsets X, Y and Z of Ω.

c) It is possible that $X \oplus (Y \cap Z) \neq (X \oplus Y) \cap (X \oplus Z)$ and $X \boxplus (Y \cap Z) \neq (X \boxplus Y) \cap (X \boxplus Z)$ for some subsets X, Y and Z of Ω.

d) It is possible that $X \oplus (Y \cup Z) \neq (X \oplus Y) \cup (X \oplus Z)$ and $X \boxplus (Y \cup Z) \neq (X \boxplus Y) \cup (X \boxplus Z)$ for some subsets X, Y and Z of Ω.

e) It is possible that $Z \cap (X \oplus Y) \neq (Z \cap X) \oplus (Z \cap Y)$ for some subsets X, Y and Z of Ω.

Proof. a) Indeed, taking $X = \{w, -w\}$, we have $\{w, -w\} \boxplus \{w, -w\} = \{w, -w\} \oplus \{w, -w\} = \varnothing$.

b) Let us take $\Omega = \{w_1, w_2, w_3, -w_1, -w_2, -w_3\}$, $Z = \{w_1, w_2, -w_1, -w_2\}$, $X = \{w_1, w_2\}$ and $Y = \{-w_1, -w_2\}$. Then $X \oplus Y = \emptyset$, $Z \cup X = Z$, $Z \cup Y = Z$, $Z \cup (X \oplus Y) = Z$, and $(Z \cup X) \oplus (Z \cup Y) = Z \oplus Z = \emptyset$. Thus, $Z \cup (X \oplus Y) \neq (Z \cup X) \oplus (Z \cup Y)$.

In a similar way, we have $X \boxplus Y = \emptyset$, $Z \cup X = Z$, $Z \cup Y = Z$, $Z \cup (X \boxplus Y) = Z$, and $(Z \cup X) \boxplus (Z \cup Y) = Z \boxplus Z = \emptyset$. Thus, $Z \cup (X \boxplus Y) \neq (Z \cup X) \boxplus (Z \cup Y)$.

c) Let us take $\Omega = \{w_1, w_2, w_3, -w_1, -w_2, -w_3\}$, $X = \{w_1, w_2, -w_1, -w_2\}$, $Z = \{w_1\}$ and $Y = \{-w_2\}$. Then $Y \cap Z = \emptyset$, $X \cap Z = Z = \{w_1\}$, $X \cap Y = Y = \{-w_2\}$, $X \boxplus (Y \cap Z) = X = \{w_1, w_2, -w_1, -w_2\}$, while $(X \boxplus Y) \cap (X \boxplus Z) = \emptyset$. Thus, $X \boxplus (Y \cap Z) \neq (X \boxplus Y) \cap (X \boxplus Z)$.

In a similar way, taking $X = \{w_1, -w_2\}$, $Z = \{w_1\}$ and $Y = \{-w_2\}$, we have $Y \cap Z = \emptyset$, $X \cap Z = Z = \{w_1\}$, $X \cap Y = Y = \{-w_2\}$, $X \oplus (Y \cap Z) = X = \{w_1, -w_2\}$, while $(X \oplus Y) \cap (X \oplus Z) = \emptyset$. Thus, $X \oplus (Y \cap Z) \neq (X \oplus Y) \cap (X \oplus Z)$.

d) Let us take $\Omega = \{w_1, w_2, w_3, -w_1, -w_2, -w_3\}$, $X = \{w_1, w_2, -w_1, -w_2\}$, $Z = \{w_1\}$ and $Y = \{-w_2\}$. Then $Y \cup Z = \{w_1, -w_2\}$, $X \boxplus Z = \{w_1, w_2, -w_2\}$, $X \boxplus Y = \{w_1, -w_1, -w_2\}$, $X \boxplus (Y \cup Z) = \{w_1, -w_2\}$, while $(X \boxplus Y) \cup (X \boxplus Z) = \{w_1, w_2, -w_1, -w_2\}$. Thus, $X \boxplus (Y \cup Z) \neq (X \boxplus Y) \cup (X \boxplus Z)$.

In a similar way, taking $X = \{-w_1, w_2\}$, $Z = \{w_1\}$ and $Y = \{-w_2\}$, we have $Y \cup Z = \{w_1, -w_2\}$, $X \oplus Z = \{w_2\}$, $X \oplus Y = \{-w_1\}$, $X \oplus (Y \cup Z) = \emptyset$, while $(X \oplus Y) \cup (X \oplus Z) = \{-w_1, w_2\}$. Thus, $X \oplus (Y \cup Z) \neq (X \oplus Y) \cup (X \oplus Z)$.

e) Let us take $\Omega = \{w_1, w_2, w_3, -w_1, -w_2, -w_3\}$, $Z = \{w_1, w_2, -w_1\}$, $X = \{w_1, w_2\}$ and $Y = \{-w_1, -w_2\}$. Then $X \boxplus Y = \emptyset$, $Z \cap X = X = \{w_1, w_2\}$, $Z \cap Y = \{-w_1\}$, $Z \cap (X \boxplus Y) = \emptyset$, and $(Z \cap X) \boxplus (Z \cap Y) = \{w_2\}$. Thus, $Z \cap (X \boxplus Y) \neq (Z \cap X) \boxplus (Z \cap Y)$.

In a similar way, we show that it is possible that $Z \cap (X \oplus Y) \neq (Z \cap X) \oplus (Z \cap Y)$.

Proposition is proved.

Proposition 3.3. $Z \cap (X \oplus Y) \subseteq (Z \cap X) \oplus (Z \cap Y)$ for all subsets X, Y and Z of Ω.

Proof. If $w \in Z \cap (X \oplus Y)$, then $w \in Z$ and $w \in X \oplus Y$. The second membership means that either ($w \in X$ and $-w \notin Y$) or ($w \in Y$ and $-w \notin X$). In the first case, $w \in Z \cap X$ and $-w \notin Z \cap Y$. In the second case, $w \in Z \cap Y$ and $-w \notin Z \cap X$. So, in both cases, $w \in (Z \cap X) \oplus (Z \cap Y)$. Thus, $Z \cap (X \oplus Y) \subseteq (Z \cap X) \oplus (Z \cap Y)$.

Proposition is proved.

4. Symmetric Hyperprobabilities

Here we study only the case with a finite number of events, i.e., when $\Omega = \{w_1, w_2, w_3, \ldots, w_n, -w_1, -w_2, -w_3, \ldots, -w_n\}$, $\Omega^+ = \{w_1, w_2, w_3, \ldots, w_n\}$ and $\Omega^- = \{-w_1, -w_2, -w_3, \ldots, -w_n\}$.

Let us consider a subset **F** of the set of subsets of Ω. By tradition, we call elements from **F** random events.

Definition 4.1. A function P from **F** to the set $\boldsymbol{R}_\omega$ of real hypernumbers is called a *symmetric hyperprobability function*, if it satisfies the following axioms:

Axiom SHP1. (*Order structure*) There is a graded with respect to Ω^+ and Ω^- involution $\alpha: \Omega \to \Omega$, i.e., α is a mapping with the following properties: α^2 is an identity mapping on Ω, $\alpha(w) = -w$ for any element w from Ω, $\alpha(\Omega^+) \supseteq \Omega^-$, and if $w \in \Omega^+$, then $\alpha(w) \notin \Omega^+$.

Axiom SHP2. (*Algebraic structure*). $\mathbf{F}^+ = \{X \in \mathbf{F};\ X \subseteq \Omega^+\}$ is a set algebra [76] that has Ω^+ as its element.

Axiom SHP3. (*Operational structure*). **F** is closed with respect to annihilation, i.e., if $X \equiv Y$ and $X \in \mathbf{F}$, then $Y \in \mathbf{F}$.

Axiom SHP4. (*Composition*) $\mathbf{F} \equiv \{X;\ X^+ \subseteq \mathbf{F}^+ \ \&\ X^- \subseteq \mathbf{F}^- \ \&\ X^+ \cap -X^{--} \equiv \varnothing \ \&\ X^- \cap -X^+ \equiv \varnothing\}$.

This axiom means that in a general case, each element from **F**, i.e., a random event, consists of two parts – one from $\mathbf{F}^+$, which is a positive random event, and another from $\mathbf{F}^-$, which is a negative random event, even though one of these parts may be empty.

Axiom SHP6. (*Normalization*) $0 \le P(A) \le 1$ for all $A \in \mathbf{F}^+$ and $P(\Omega^+) = 1$.

Axiom SHP7. (*Finite additivity*)

$P\,(A \cup B) = P\,(A) + P\,(B)$

for all random events $A, B \in \mathbf{F}$ such that

$A \cap B \equiv \emptyset$

Axiom SHP8. (*Symmetry*) For any random event A from **F**, $P\,(A) = -\,P\,(-A)$.

Note that this axiom supports and is supported by intuition of physicists with respect to negative probability. For instance, Dirac [64] compared negative probability to (the probability of) a negative sum of money, i.e., to a debt. Indeed, debts annihilate (at least, to some extent) those amounts of money that people have. In a similar way, Belinskii [27] writes, “a negative probability *reduces* the probability for events corresponding to it and *increases* the probability for opposite events.”

Axioms SHP6 and SHP8 imply the following results.

Proposition 4.1. (*Non-positivity*) $P(A) \le 0$ for all $A \in \mathbf{F}^-$.

Proposition 4.2. (*Symmetry*) $P(-A) = -\,P(A)$ for all $A \in \mathbf{F}$.

Corollary 4.1. $P(\Omega^-) = -1$.

We can see that Axioms SHP1-SHP8 establish a connection between symmetric hyperprobabilities and signed hypermeasures [65] in the same way as Kolmogorov axioms determine a connection between conventional probabilities and measures [66].

Proposition 4.3. (*Adequacy*) If $A \equiv B$ and $A \in \mathbf{F}$, then $P(A) = P(B)$.

Proof. If $A \equiv B$, then $A = C \cup H \cup -H$, $B = C \cup K \cup -K$ and $C \cap H = C \cap -H = C \cap K = C \cap -K = \emptyset$. By Axiom SHP7, $P(A) = P(C) + P(H) + P(-H)$ and by Axiom SHP8, $P(H) + P(-H) = 0$. In a similar way, by Axiom SHP7, $P(B) = P(C) + P(K) + P(-K)$ and by Proposition 4.2, $P(K) + P(-K) = 0$. Consequently, $P(A) = P(B)$.

Proposition is proved.

For instance, $P(\{w, -w\}) = P(\emptyset) = 0$ because $\{w, -w\} \equiv \emptyset$.

As α is an involution of the whole space, we have the following result.

Lemma 4.1. α is a one-to-one mapping and $|\ \Omega^+\ | = |\Omega^-|$.

Corollary 4.2. (*Domain symmetry*) $w \in \Omega^+$ if and only if $-w \in \Omega^-$.

Corollary 4.3. (*Element symmetry*) $-\,(-\,w) = w$ for any element w from Ω.

Corollary 4.4. (*Event symmetry*) $-(-A) = A$ for any event A from 2^Ω.

Lemma 4.2. $\alpha(w) \ne w$ for any element w from Ω.

Proof. For any $w \in \Omega^+$ this is true by Axiom SHP1. If for some $w \in \Omega^-$, we have $\alpha(w) = w$, then $\alpha(v) = w$ for some element v from Ω^+ because by Axiom SHP1, α is a projection of Ω^+ onto Ω^-. Consequently, we have

$\alpha(\alpha(v)) = \alpha(w) = w$

However, α is an involution, and we have $\alpha(\alpha(v)) = v$. This results in the equality

$v = w$.

Consequently, we have $\alpha(v) = v$. This contradicts Axiom SHP1 because $v \in \Omega^+$.

Lemma is proved.

Lemma 4.3. $\Omega^+ \cap \Omega^- \equiv \emptyset$.

Proof. Let $w \in \Omega^+ \cap \Omega^-$. Then by Axiom EHP1, we have $-w \in \Omega \setminus \Omega^+ = \Omega^- \setminus \Omega^+$ as $\Omega = \Omega^- \cup \Omega^+$. Thus, we can define $\Omega_0^- = \Omega^- \setminus \{w\}$ and take $\Omega = \Omega_0^- \cup \Omega^+$. However, this contradicts irreducibility of Ω^-.

Proposition is proved by contradiction.

Corollary 4.5. For any element w from Ω, $w \in \Omega^+$ if and only if $-w \in \Omega^-$.

An important property of the basic set-theoretical operations is that it is possible to perform operations separately on positive and negative components of sets and then to combine these results.

Proposition 4.4. $A \cup B = (A^+ \cup B^+) \cup (A^- \cup B^-)$ for any subsets A and B of Ω.

Indeed, as $A \equiv A^+ \cup A^-$ and $B \equiv B^+ \cup B^-$, we have

$A \cup B = (A^+ \cup A^-) \cup (B^+ \cup B^-) = (A^+ \cup B^+) \cup (A^- \cup B^-)$

Proposition 4.5. $A \cap B \equiv (A^+ \cap B^+) \oplus (A^- \cap B^-)$ for any subsets A and B of Ω.

Indeed, as $A \equiv A^+ \cup A^-$ and $B \equiv B^+ \cup B^-$, we have

$A \cap B \equiv (A^+ \cup A^-) \cap (B^+ \cap B^-) \equiv$

$(A^+ \cap B^+) \cup (A^+ \cap B^-) \cup (A^- \cap B^+) \cup (A^- \cap B^-) \equiv (A^+ \cap B^+) \oplus (A^- \cap B^-)$

because $(A^+ \cap B^-) \equiv \emptyset$ and $(A^- \cap B^+) \equiv \emptyset$.

In a similar way, we prove the following results.

Proposition 4.6. $A \setminus B \equiv (A^+ \setminus B^+) \oplus (A^- \setminus B^-)$ for any subsets A and B of Ω.

Proposition 4.7. $X \equiv X^+ \oplus X^- = X^+ \cup X^-$ for any set X from **F**.

Indeed, as $X \subseteq \Omega$, $X^+ = X \cap \Omega^+$, $X^- = X \cap \Omega^-$, and $\Omega = \Omega^+ \cup \Omega^-$, we have $X = X^+ \cup X^-$. In addition, $X^+ \oplus X^- = X^+ \cup X^-$ because by Proposition 4.4, we have $X^+ \cap X^- = \emptyset$.

Proposition 4.8. $A \oplus B \equiv (A^+ \oplus B^+) \oplus (A^- \oplus B^-)$ for any sets X and Y from **F**.

Indeed, as by Proposition 4.7, $A \equiv A^+ \oplus A^-$ and $B \equiv B^+ \oplus B^-$, we have

$A \oplus B \equiv (A^+ \oplus A^-) \oplus (B^+ \oplus B^-) \equiv (A^+ \oplus B^+) \oplus (A^- \oplus B^-)$

because by Proposition 3.1, operation $\oplus$ is commutative and associative.

Theorem 4.1. The system of Axioms SHP1- SHP8 is consistent.

Indeed, we can take $\Omega = \{w, -w\}$, $\Omega^+ = \{w\}$, $\Omega^- = \{-w\}$, **F** $= \{\emptyset, \{w, -w\}, \{w\}, \{-w\}\}$ and assign $P(\emptyset) = 0$, $P(\{w, -w\}= 0$, $P(\{w\}) = 1$, and $P(\{-w\}) = -1$. Then it is easy to check that P satisfies all Axioms SHP1- SHP8.

Definition 4.2. a) The triad (Ω, **F**, P) is called a *symmetric hyperprobability space*.

b) If $A \in$ **F**, then the number $P(A)$ is called the *symmetric hyperprobability* of the event A.

Let us obtain some properties of symmetric hyperprobability.

Proposition 4.9. $P(\varnothing) = 0$.

Indeed, by Axiom SHP7, we have $P(A \cup \varnothing) = P(A) + P(\varnothing)$. Thus, $P(\varnothing) = P(A \cup \varnothing) - P(A) = P(A) - P(A) = 0$.

Proposition 4.9 has the following interpretation. In each trial (experiment), something happens. Therefore, the symmetric hyperprobability that nothing happens is equal to zero.

Proposition 3.1 and Axioms SHP4 and SHP8 imply the following result.

Proposition 4.10. $P(X \oplus Y) = P(X \boxplus Y) = P(X \cup Y)$ for any two random events X and Y from **F**.

Proposition 4.11. $0 \geq P(C) \geq -1$ for all $C \in \mathbf{F}^-$.

Proof. By Proposition 4.2, $P(C) = - P(A)$ for some random event $A \in \mathbf{F}^+$ and $P(\Omega^-) = -P(\Omega^+)$. By Axioms SHP6, $0 \leq P(A) \leq 1$. Thus, $0 \geq P(C) \geq -1$.

Proposition is proved.

The symmetric hyperprobability of a random event is composed from two components as the following theorem shows.

Proposition 4.12. $P(A) = P(A^+) - P(- A^-) = P(A^+) + P(A^-)$ for any random event $A \in \mathbf{F}$.

Proof. As $A = A^+ \cup A^-$, by Axiom SIP7, $P(A) = P(A^+) + P(- A^-)$ because by Lemma 2.6, $A^+ \cap A^- = Ø$ for any subset A of Ω. By Proposition 2.3, $P(A^-) = - P(- A^-)$. Consequently, $P(A) = P(A^+) - P(- A^-)$.

Propositions 4.11 and 4.12 imply the following result.

Theorem 4.2. $-1 \leq P(A) \leq 1$ for all $A \in \mathbf{F}$.

Indeed, by Propositions 4. 13, $P(A) = P(A^+) - P(- A^-)$ and $P(- A^-) \geq 0$. Thus, $P(A) \leq P(A^+) \leq P(\Omega^+)$. In addition, $P(\Omega^-) \leq P(A^-) \leq P(A)$. Thus, $P(\Omega^-) \leq P(A) \leq P(\Omega^+)$ and $-1 \leq P(A) \leq 1$ because $P(\Omega^-) = -1$ and $P(\Omega^+) = 1$.

Proposition 4.13. $P(\Omega) = 0$.

Indeed, $P(\Omega) = P(\Omega^-) + P(\Omega^+)$ and $P(\Omega^-) = -P(\Omega^+)$.

Proposition 4.13 has the following interpretation. The set Ω cannot exist (be stable) due to annihilation. So, its symmetric hyperprobability, or more adequately, the symmetric hyperprobability of its existence, is equal to zero.

Proposition 4.14. If $A = \{w_1, w_2, w_3, \dots, w_k\}$ belongs to **F** and all $w_1, w_2, w_3, \dots, w_k$ belongs to **F**, then $P(A) = P(w_1) + P(w_2) + P(w_3) + \dots + P(w_k)$.

Proposition 4.14 directly follows from Axiom SHP7 because A is a set of elementary random events.

Proposition 4.15. Any symmetric hyperprobability function P is monotone on $\mathbf{F}^+$, i.e., if $A \subseteq B$ and $A, B \in \mathbf{F}^+$, then $P(A) \leq P(B)$, and is antimonotone on $\mathbf{F}^-$, i.e., if $H \subseteq K$ and $H, K \in \mathbf{F}^-$, then $P(H) \geq P(K)$.

Proof. Let us consider two random events A and B from $\mathbf{F}^+$, such that $A \subseteq B$. In this case, $B = A \cup C$ for some random event C from $\mathbf{F}^+$ where $A \cap C = Ø$. By Axiom SHP7, $P(B) = P(A) + P(C)$ and by Proposition 4.1, both $P(A)$ and $P(C)$ are positive hypernumbers. Thus, $P(A) \leq P(B)$.

Let us consider two random events H and K from $\mathbf{F}^-$, such that $H \subseteq K$. In this case, $K = H \cup G$ for some random event G from $\mathbf{F}^-$ where $H \cap G = Ø$. By Axiom SHP7, $P(K) = P(H) + P(G)$ and by Proposition 4.2, both $P(H)$ and $P(G)$ are negative hypernumbers. Thus, $P(K) \leq P(H)$.

Proposition is proved.

The set ***R*** of all real numbers is a subset of set $\boldsymbol{R}_\omega$ of all real hypernumbers. Let us take the restriction P_R of a hyperprobability function *P* on the set ***R***. Direct comparison of axioms SHP1 – SHP3 and axioms SIP1 – SIP8, which characterize symmetric probability functions [67], gives us the following result.

Theorem 4.3. The function P_R satisfies axioms SIP1 – SIP8 and thus, is a symmetric probability function.

5. Conclusion

We have developed a mathematical theory of hyperprobability aiming of extending probabilistic reasoning in various areas from quantum physics and quantum computation through cognition, economics and finance. The obtained results allow rigorous application of hyperprobabilities that take both positive and negative values in these and other areas.

The obtained results also open new problems and directions for further research in hyperprobability theory.

Limit properties are important for probability and its applications [68].

1. Find what limit properties of probabilities remain valid for hyperprobabilities.

Probabilities, that take value larger than 1, are used in physics, biology and mathematical finance [15, 19, 69, 70]. This brings us to the following problem.

2. Study hyperprobabilities that take values larger than 1.

In probability theory, conditional probabilities play an important role. Some researchers even suggest to take the concept of conditional probability as the basic for probability theory [71-74]. The same is true for hyperprobabilities [75]. This brings us to the following problem.

3. Study conditional symmetric hyperprobabilities.

Shannon's information theory is based on the conventional concept of probability [77]. Emergence of negative probabilities and hyperprobabilities bring us to the following problems.

4. Develop information theory based on symmetric probabilities including negative probabilities.

5. Develop information theory based on hyperprobabilities.

6. Conflicts of Interest

The author(s) report(s) no conflict(s) of interest(s). The author along are responsible for content and writing of the paper.

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