UPI Journal of Mathematics and Biostatistics 2018; 1(1): 22-31

**Research Article** ISSN: 2581-4672 **Open Access**

UPI JOURNAL OF MATHEMATICS AND BIOSTATISTICS

Journal Home Page: https://uniquepubinternational.com/upi-journals/upi-journal-of-mathematics-and-biostatistics-upi-jmb/

# Convexity on Fractal Sets

**Teodoro Lara[1*], Nelson Merentes[2], Edgar Rosales[1], Rainier Sanchez[3]**

[1]Departamento De FiSica Y Matematicas, Universidad De Los Andes, Trujillo, Venezuela.

[2]Universidad Central de Venezuela. Escuela de matematicas, Caracas, Venezuela.

[3]Universidad Politecnica Territorial del Oeste de Sucre, Cumana, Venezuela.

**How to cite:** Lara T, Merentes N, Rosales E, Sanchez R. Convexity on Fractal Sets. UPI Journal of Mathematics and Biostatistics 2018; 1(1): 22-31.

**Article history:**
Received : 15-03-2018, Accepted: 24-03-2018,
Published: 25-03-2018

**Correspondence to**: Lara T, Departamento De FiSica Y Matematicas, Universidad De Los Andes, Trujillo, Venezuela. Email: tlara@ula.ve

**Funding:** This research has been partially supported by Central Bank of Venezuela, 2010, Mathematics Subject Classification. 26A51, 39B62, 26D07, 26D15.

**UPI Journal of Mathematics and Biostatistics 2018; 1(1): 22-31**

# Convexity on Fractal Sets

Teodoro Lara, Nelson Meretes, Edgar Rosales, Rainier Sanchez

**Abstract:** In this research we address convexity of functions on fractal sets; these type of functions are named generalized convex functions on fractal sets. Our results deal with algebraic properties, boundedness and lipschitzity of this class of functions.

**Key words:** Convexity, Fractal set, Mass function, α-type set.

## Introduction

At the present time convex functions play a key role in different areas of mathematics like functional analysis, complex analysis and differential equations among others. They find applications in disciplines as economy, biology, physics, and optimization [3, 7, 15].

We recall that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on $I$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

for any $x, y \in I$, $t \in [0, 1]$, $I$ being an interval [5, 8, 10, 11, 14] and references therein. Different types of convexity have been object of study and research by many authors for quite long period of time, we may mention strongly convex functions, midconvex functions, $s$ and $m$-convex functions only to name few of them [2, 5, 9, 17] and reference therein.

Recently, the theory of fractal sets has called the interest of different class of researchers whose have found many applications, we may cite [1, 4, 6, 9, 18] and many more.

In this paper we set some algebraic and topological properties of convex functions defined on fractal sets, show that these functions in this context are bounded and Lipschitz on any compact interval $[a, b]$ in the corresponding domain; some few inequalities are established as well. Our results are inspired basically on similar results for convex functions defined on real intervals

Recently, [18] studies local fractional functions on fractal sets and introduces the theory of fractal sets in the following way. For $0 < \alpha \leq 1$, the so called $\alpha$-type sets are defined

$$\begin{aligned}
\mathbb{Z}^{\alpha} &= \{0^{\alpha}, \pm 1^{\alpha}, \pm 2^{\alpha}, \cdots, \pm n^{\alpha}, \cdots\} \text{ (integers numbers of } \alpha\text{-type).}\\
\mathbb{Q}^{\alpha} &= \left\{m^{\alpha} = \left(\frac{p}{q}\right)^{\alpha} : p, q \in \mathbb{Z}, q \neq 0\right\} \text{ (rational numbers of } \alpha\text{-type).}\\
\mathbb{J}^{\alpha} &= \left\{m^{\alpha} \neq \left(\frac{p}{q}\right)^{\alpha} : p, q \in \mathbb{Z}, q \neq 0\right\} \text{ (irrational numbers of } \alpha\text{-type).}\\
\mathbb{R}^{\alpha} &= \mathbb{Q}^{\alpha} \cup \mathbb{J}^{\alpha} = \{a^{\alpha} : a \in \mathbb{R}\} \text{ (real numbers of } \alpha\text{-type).}
\end{aligned}$$

Any subset of $\mathbb{R}^{\alpha}$ will be called a fractal set [4, 18, 19]. If $a^{\alpha}$, $b^{\alpha}$ and $c^{\alpha}$ are in $\mathbb{R}^{\alpha}$ then [13, 18, 19], the following properties take place,

(1) $a^{\alpha} + b^{\alpha}$ y $a^{\alpha} b^{\alpha}$ are in $\mathbb{R}^{\alpha}$ as well.
(2) $a^{\alpha} + b^{\alpha} = b^{\alpha} + a^{\alpha} = (a + b)^{\alpha} = (b + a)^{\alpha}$.
(3) $a^{\alpha} - b^{\alpha} = (a - b)^{\alpha}$.
(4) $a^{\alpha} + (b^{\alpha} + c^{\alpha}) = (a^{\alpha} + b^{\alpha}) + c^{\alpha}$.
(5) $a^{\alpha} b^{\alpha} = b^{\alpha} a^{\alpha} = (ab)^{\alpha} = (ba)^{\alpha}$.
(6) $a^{\alpha}(b^{\alpha} c^{\alpha}) = (a^{\alpha} b^{\alpha}) c^{\alpha}$.
(7) $a^{\alpha}(b^{\alpha} + c^{\alpha}) = a^{\alpha} b^{\alpha} + a^{\alpha} c^{\alpha}$.
(8) $a^{\alpha} + 0^{\alpha} = 0^{\alpha} + a^{\alpha} = a^{\alpha}$ y $a^{\alpha} 1^{\alpha} = 1^{\alpha} a^{\alpha} = a^{\alpha}$.

**Remark 1.** *The above listed properties come out as consequence of properties of mass function* $\gamma^{\alpha}(F, a, b)$ *where* $F$ *is a fractal set,* $a$ *and* $b$ *are real numbers; details may be seen in* [13]. *In particular, item* (2) *follows from the local fractional integral and its properties given in* [18, 19].

**Definition 2.** *(*[9]*) Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}^{\alpha}$ *be a function,* $f$ *is called generalized convex on* $I$ *if for any* $x, y \in I$ *and* $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq t^{\alpha} f(x) + (1 - t)^{\alpha} f(y).$$

The set of all generalized convex functions defined on $I$ will be denoted as $GC_{\alpha}(I)$. In other words,

$$GC_{\alpha}(I) = \{f : I \subseteq \mathbb{R} \to \mathbb{R}^{\alpha} : f \text{ is generalized convex on } I\}.$$

In a similar manner a function $g : I \subseteq \mathbb{R} \to \mathbb{R}^{\alpha}$ is named generalized concave on $I$ if for any $x, y \in I$, $t \in [0, 1]$

$$f(tx + (1 - t)y) \geq t^{\alpha} f(x) + (1 - t)^{\alpha} f(y).$$

The function

$$f(x) = x^{2\alpha},\ x \geq 0,\ 0 < \alpha \leq 1$$

given in [9, 16], is shown to be generalized convex. More over, the case $\alpha = 1$ corresponds to a known convex functions while if $\alpha = \frac{1}{4}$ $f$ becomes a concave function (therefore, non convex ) and illustrates that the class of generalized convex functions is different from the set of convex ones.

## Basic rules

In this section we show some basic results involving generalized convex functions, particularly concerning addition, product, and composition, the same are inspired in similar results for convex functions on the real line [8, 10, 14].

**Proposition 3.** *Si* $f, g \in GC_\alpha(I)$ *then*

(1) $f + g \in GC_\alpha(I)$.
(2) *If* $k \in \mathbb{R}^\alpha$, $kf \in GC_\alpha(I)$.

**Definition 4.** *(*[12, 15]*) Two given functions* $f$ *and* $g$ *are said to be similarly ordered if* $(f(x) - f(y))(g(x) - g(y)) \geq 0$, *for any* $x, y \in Dom(f) = Dom(g)$.

**Proposition 5.** *If* $f$ *and* $g$ *are similarly ordered and* $f, g \in GC_\alpha(I)$, *then* $fg \in GC_\alpha(I)$.

*Proof.* Let $x, y \in I$ y $t \in [0,1]$ so,

$$\begin{aligned}
(fg)(tx + (1-t)y) &= f(tx + (1-t)y)g(tx + (1-t)y) \\
&\leq t^\alpha \left(t^\alpha (fg)(x) + (1-t)^\alpha (fg)(x)\right) + (1-t)^\alpha (t^\alpha (fg)(y) \\
&\quad +(1-t)^\alpha (fg)(y)) \\
&= t^\alpha (fg)(x) + (1-t)^\alpha (fg)(y).
\end{aligned}$$

□

**Proposition 6.** *Let* $I'$ *be an interval,* $f \in GC_\alpha(I')$, $g : I \to I'$, *with* $f$ *be non decreasing and* $g$ *convex functions respectively, ,then* $f \circ g \in GC_\alpha(I)$.

*Proof.* Let $x, y \in I$, $t \in [0,1]$, since $g$ is convex

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y).$$

On the other hand, being $f$ non decreasing in $GC_\alpha(I')$

$$\begin{aligned}
(f \circ g)(tx + (1-t)y) &= f(g(tx + (1-t)y)) \\
&\leq f\left(tg(x) + (1-t)g(y)\right) \\
&\leq t^\alpha (f \circ g)(x) + (1-t)^\alpha (f \circ g)(y).
\end{aligned}$$

□

**Proposition 7.** *If* $\left\{f_n\right\}_{n=1}^{+\infty}$ *is a sequence in* $GC_\alpha(I)$ *pointwise converging to function* $f : I \to \mathbb{R}^\alpha$, *then* $f \in GC_\alpha(I)$.

*Proof.* For $x, y \in I$, $t \in [0,1]$ and $\lim_{n\to\infty} f_n(x) = f(x)$,

$$\begin{aligned} f(tx+(1-t)y) &= \lim_{n\to\infty} f_n(tx+(1-t)y) \\ &\leq \lim_{n\to\infty} \left[t^\alpha f_n(x) + (1-t)^\alpha f_n(y)\right] \\ &= t^\alpha f(x) + (1-t)^\alpha f(y), \end{aligned}$$

and we are done. □

## Main results

This section is devoted to establish and show our main results, among them some characterizations and boundedness of generalized convex functions on compact intervals. Incoming two results are similar to those given in [10] and references therein for convex functions

**Theorem 8.** *For $f \in GC_\alpha(I)$; $x, y, z \in I$ y $t \in [0,1]$, the following statements are equivalent*

(1) $f \in GC_\alpha(I)$.

(2) $f(x + t(y-x)) \leq f(x) + t^\alpha (f(y) - f(x))$.

(3) $f(sx + ty) \leq s^\alpha f(x) + t^\alpha f(y)$, *and* $s + t = 1$.

(4) $\begin{vmatrix} x^\alpha & -f(x) & 1 \\ y^\alpha & -f(y) & 1 \\ z^\alpha & -f(z) & 1 \end{vmatrix} = (y-z)^\alpha f(x) + (z-x)^\alpha f(y) - (y-x)^\alpha f(z) \geq 0.$

(5)

$$f(x,y,z) = \frac{f(x)}{(z-x)^\alpha (y-x)^\alpha} + \frac{f(y)}{(y-z)^\alpha (y-x)^\alpha} - \frac{f(z)}{(y-z)^\alpha (z-x)^\alpha} \geq 0. \tag{1}$$

*Proof.* (1) ⇒ (2). Is immediate.

(2) ⇒ (3). Since $s + t = 1$, $s = 1 - t$ and by using (2)

$$\begin{aligned} f(x+t(y-x)) &= f(ty+(1-t)x) \\ &\leq f(x) + t^\alpha (f(y) - f(x)) \\ &= s^\alpha f(x) + t^\alpha f(y). \end{aligned}$$

(3) ⇒ (4). For $x, y, z \in I$ with $x < z < y$ there is $t \in [0,1]$ such that $z = tx + (1-t)y$, hence $t = \frac{y-z}{y-x}$ and $1 - t = \frac{z-x}{y-x}$. Therefore,

$$\begin{aligned} f(z) &= f(tx+(1-t)y) \\ &= f\left(\frac{y-z}{y-x}x + \frac{z-x}{y-x}y\right) \\ &\leq \left(\frac{y-z}{y-x}\right)^\alpha f(x) + \left(\frac{z-x}{y-x}\right)^\alpha f(y). \end{aligned}$$

or better,

$$(y-z)^{\alpha}f(x)+(z-x)^{\alpha}f(y)-(y-x)^{\alpha}f(z)\geq 0.$$

But

$$\begin{vmatrix} x^{\alpha} & -f(x) & 1 \\ y^{\alpha} & -f(y) & 1 \\ z^{\alpha} & -f(z) & 1 \end{vmatrix} = (y-z)^{\alpha}f(x)+(z-x)^{\alpha}f(y)-(y-x)^{\alpha}f(z)\geq 0.$$

$(4)\Rightarrow(5)$. Is trivial

$(5)\Rightarrow(1)$. Multiplying (1) by $(y-z)^{\alpha}(z-x)^{\alpha}(y-x)^{\alpha}\geq 0$, we obtain

$$(y-z)^{\alpha}f(x)+(z-x)^{\alpha}f(y)-(y-x)^{\alpha}f(z)\geq 0,$$

that is,

$$f(z) \;\leq\; \left(\frac{y-z}{y-x}\right)^{\alpha} f(x)+\left(\frac{z-x}{y-x}\right)^{\alpha} f(y)$$

but by hypothesis $x,y,z\in I$ , $x<z<y$, so we may find $t\in[0,1]$ in such a way that $z=tx+(1-t)y$, which in turn means $t=\frac{y-z}{y-x}$ and $1-t=\frac{z-x}{y-x}$. Whence,

$$f(tx+(1-t)y) \;\leq\; t^{\alpha}f(x)+(1-t)^{\alpha}f(y).$$

□

A consequence of foregoing theorem is

**Proposition 9.** *Let $f\in GC_{\alpha}(I)$; $x,y,z\in I$, with $x<y<z$. Then*

$$\frac{f(y)-f(x)}{(y-x)^{\alpha}}\leq\frac{f(z)-f(x)}{(z-x)^{\alpha}}\leq\frac{f(z)-f(y)}{(z-y)^{\alpha}}.$$

*Proof.* For $x<y<z$, is possible to find $t\in[0,1]$ for which $y=(1-t)x+tz$, hence

$$f(y)=f\left(x+\frac{y-x}{z-x}(z-x)\right)\leq f(x)+\left(\frac{y-x}{z-x}\right)^{\alpha}(f(z)-f(x)),$$

but then

$$\frac{f(y)-f(x)}{(y-x)^{\alpha}}\leq\frac{f(z)-f(x)}{(z-x)^{\alpha}}. \tag{2}$$

In the same token, and since $z=x+\frac{z-x}{z-y}(z-y)$,

$$\frac{f(z)-f(x)}{(z-x)^{\alpha}}\leq\frac{f(z)-f(y)}{(z-y)^{\alpha}}. \tag{3}$$

Conclusion follows from (2) and (3). □

**Corollary 10.** *If $f\in GC_{\alpha}(I)$ then for any $x\in I$ the function $\bar{f}:I-\{x\}\to\mathbb{R}^{\alpha}$ given by $\bar{f}(u)=\frac{f(x)-f(u)}{(x-u)^{\alpha}}$ is increasing.*

A subset $A \subset \mathbb{R}^\alpha$ is called generalized convex if for all $x^\alpha, y^\alpha \in A$, the number $z = (1-t)^\alpha x^\alpha + t^\alpha y^\alpha \in A$, for arbitrary $t \in [0,1]$. In this context a function $g : I \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ is called affine function if there exist $a, b \in \mathbb{R}$ such that for any $x \in I$, $g(x) = a^\alpha x^\alpha + b^\alpha$. An immediate consequence of this fact is that any affine function is both, generalized convex and generalized concave function.

Last part of this research is devoted to boundedness and lipschitzity of generalized convex functions, results here are inspired basically in results from[11].

**Proposition 11.** *If $f \in GC_\alpha(I)$ then for every compact subinterval $A \subseteq I$ and arbitrary generalized affine function $g$ defined on $I$,* $\sup(f+g)$ *is attained at one of the endpoints of $A$.*

*Proof.* Let $A$ be given as $A = [x, y]$, $x, y \in I$, therefore any $z \in A$ can be written as $z = (1-t)x + ty$ for some $t \in [0,1]$, hence by using properties of elements in $\mathbb{R}^\alpha$,

$$\begin{aligned}(f+g)(z) &= f((1-t)x+ty) + g((1-t)x+ty)\\ &\leq (1-t)^\alpha f(x) + t^\alpha f(y) + (1-t)^\alpha g(x) + t^\alpha g(y)\\ &= f(x) + g(x) - t^\alpha \left[(f(x)+g(x)) - (f(y)+g(y))\right],\end{aligned}$$

or better

$$\begin{aligned}\sup_{z\in A}(f+g)(z) &= \begin{cases} f(x)+g(x), & \text{si } f(x)+g(x) \geq f(y)+g(y)\\ f(y)+g(y), & \text{si } f(x)+g(x) < f(y)+g(y)\end{cases}\\ &= \max\{f(x)+g(x), f(y)+g(y)\}.\end{aligned}$$

□

**Corollary 12.** *If $f \in GC_\alpha(I)$, then $f$ is bounded on any compact interval $[a,b] \subseteq I$.*

*Proof.* For any $x \in [a,b]$ there exists $t \in [0,1]$ such that $x = (1-t)a + tb$ and from foregoing Theorem and hypothesis we are able to show that $f(x) \leq \max\{f(a), f(b)\}$ so $f$ is upper bounded on $[a,b]$. On the other hand, for arbitrary $x \in [a,b]$ it is possible to write it as $x = \frac{a+b}{2} + t$, for some $t$ such that $|t| \leq \frac{b-a}{2}$, but then

$$\begin{aligned}f(x) = f\left(\frac{a+b}{2}+t\right) &\geq 2^\alpha f\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}-t\right)\\ &\geq 2^\alpha f\left(\frac{a+b}{2}\right) - \max\{f(a), f(b)\}\\ &\equiv k,\end{aligned}$$

and $f$ is lower bounded as well. □

Now we introduce the following

**Definition 13.** *A function $f : I \subset \mathbb{R} \to \mathbb{R}^\alpha$ is named generalized Lipschitz if there exists a constant $L^\alpha \geq 0^\alpha$ in such a way that $|f(x) - f(y)| \leq L^\alpha |x-y|^\alpha$, for any $x, y \in I$.*

The following result shows up,

**Proposition 14.** *If $f \in GC_\alpha(I)$ then it is Lipschitz on any $[a,b] \subset int(I)$.*

*Proof.* Let $\epsilon > 0$ with $a+\epsilon, b-\epsilon \in int(I)$, $M^\alpha$ and $m^\alpha$ be the sup and inf of $f$ on $[a-\epsilon, b-\epsilon]$ respectively. For any $x, y \in [a,b]$, say $x < y$, the number $z = y + \epsilon \in [a+\epsilon, b+\epsilon]$ and $x < y < z$, hence there is $t \in [0,1]$ for which $y = tz + (1-t)x$, actually, $t = \frac{y-x}{y-x+\epsilon}$ and by hypothesis

$$f(y) \leq t^\alpha f(z) + (1-t)^\alpha f(x),$$

or better

$$\begin{aligned} f(x) - f(y) &\leq t^\alpha (f(z) - f(y)) \\ &= \frac{(y-x)^\alpha}{\epsilon^\alpha + (y-x)^\alpha}(f(z) - f(y)) \\ &\leq \frac{|y-x|^\alpha}{\varepsilon^\alpha}|M^\alpha - m^\alpha| \\ &= L^\alpha |y-x|^\alpha, \end{aligned}$$

and $L^\alpha = \frac{|M^\alpha - m^\alpha|^\alpha}{\epsilon^\alpha}$. Proof concludes by switching the roles of $x$ and $y$. ☐

**Theorem 15.** *For $f \in GC_\alpha(I)$ and $x, y, z \in I$ arbitraries,*

$$\begin{aligned} \frac{f(x)+f(y)+f(z)}{3^\alpha} + f\Big(\frac{x+y+z}{3}\Big) &\geq \frac{2^\alpha}{3^\alpha}\Big[f\Big(\frac{x+y}{2}\Big) \\ &\quad + f\Big(\frac{y+z}{2}\Big) + f\Big(\frac{x+z}{2}\Big)\Big]. \end{aligned}$$

*Proof.* Without loss of generality we may assume $x \leq y \leq z$. So, for $y \leq \frac{x+y+z}{3}$,

$$\frac{x+y+z}{3} \leq \frac{x+z}{2} \leq z \text{ and } \frac{x+y+z}{3} \leq \frac{y+z}{2} \leq z.$$

Hence there are $s, t \in [0,1]$ such that

$$\frac{x+z}{2} = s\frac{x+y+z}{3} + (1-s)z \text{ and } \frac{y+z}{2} = t\frac{x+y+z}{3} + (1-t)z.$$

But then $(x+y+2z)(s+t-\frac{3}{2}) = 0$. In other words, $(x+y+2z) = 0$ or $(s+t-\frac{3}{2}) = 0$.

If $(x+y+2z) = 0$, $x = y = z$ and because $f \in GC_\alpha(I)$, the following three inequalities are satisfied

(1) $f\left(\frac{x+z}{2}\right) \leq s^\alpha f\left(\frac{x+y+z}{3}\right) + (1-s)^\alpha f(z)$,

(2) $f\left(\frac{y+z}{2}\right) \leq t^\alpha f\left(\frac{x+y+z}{3}\right) + (1-t)^\alpha f(z)$,

(3) $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2^\alpha}$.

Actually in the case $x+y+2z = 0$ the result holds and is obtained just summing up inequalities (1)–(3).

For $s+t-\frac{3}{2}=0$, we proceed as above and get

$$\begin{aligned} f\Big(\frac{x+z}{2}\Big)+f\Big(\frac{y+z}{2}\Big)+f\Big(\frac{x+y}{2}\Big) &\leq (s+t)^{\alpha} f\Big(\frac{x+y+z}{3}\Big)+2^{\alpha} f(z) \\ &\quad -(s+t)^{\alpha} f(z)+\frac{f(x)}{2^{\alpha}}+\frac{f(y)}{2^{\alpha}} \\ &\leq \frac{3^{\alpha}}{2^{\alpha}} f\Big(\frac{x+y+z}{3}\Big) \\ &\quad +\frac{f(x)+f(y)+f(z)}{2^{\alpha}}. \end{aligned}$$

The case $\frac{x+y+z}{3} \leq y$ runs in a similar fashion. $\square$

## Conflict of interest

Authors declared that there are no conflicts of interests.

## Funding source

This research is partially supported by Central Bank of Venezuela, 2010, Mathematics Subject Classification.

26A51, 39B62, 26D07, 26D15.

## References

1. Babakhani A, Dastardar-Gejji. On calculus of local fractional derivatives. Journal of Mathematical Analysis and Applications 2002; 270(1): 66-79.

2. Beesack P, Peˇcari´c J. On Jensen's inequality for convex function. Journal of Mathematical Analysis and Applications 1985; 110(1): 536-552.

3. Boyd S, Vanderberghe L. Convex Optimization. Cambridge University Press, New York, NY, 2009.

4. Guang-Sheng C. A generalized Young inequality and some new results on fractal space. Advances in Computational Mathematics and its Applications 2002; 1(1): 1-7.

5. Gilanyi A, Pales Z. Convex functions of higher order. Mathematical Inequalities & Applications 2008; 11(2): 271-282.

6. Golmankhaneh A, Baleann D. On a new measure on fractals. Journal of Inequalities and Applications 2013; 522(1): 1-9.

7. Grinblatt M, Linnainmaa J. Jensen's inequality, parameter uncertainty and multiperiod incestment. Review of Asset Pricing Studies 2011; 1(1): 1-34.

8. Hiriart-Urruty JB, Lemaré chal C. Fundamentals of Convex Analysis. Springer-Verlag, Berlin-Heidelberg, 2001.

9. Huixia M, Xin S, Dongyan Y. Generalized Convex Functions on Fractal Sets and Two Related Inequalities. Abstract and Applied Analysis 2014; 2014: 636751.

10. Merentes N, Rivas S. The Development of the Concept of Convex Function (in spanish), XXVI Escuela Venezolana de Mateḿ aticas, Venezuela, 2013.

11. Niculescu C, Persson LE. Convex functions and their applications. Springer Science + Business Media, Inc. 2006.

12. Noor M, Noor K, Awan M. Generalized convexity and integral inequalities. Applied Mathematics & Information Sciences 2015; 9(1): 233-243.

13. Parvate A, Gangal A. Calculus on fractal subsets of real line I: formulation. arXiv:math-ph/0310047.

14. Roberts AW, Varberg DE. Convex Functions. Academic Press, NY, 1973.

15. Pecarí c J, Proschan F, Tong YL. Convex Functions, partial orderings and statistical applications , Academic Press, New York, (1992).

16. Set JE, Tomar M. New Inequalities of Hermite-Hadamard Type for Generalized Convex Functions with Applications. Mathematics and Informatics 2016; 31(2): 383-397.

17. Toader G. Some generalizations of the convexity. Proceedings of the Colloquium on Approximation and Optimization, Cluj-Napoca (Romania), 1984, pp. 329-338.

18. Yang X. Advanced Local Fractional Calculus and Its Applications. World Science, New York, NY, 2012.

19. Jun YX. Expression of Generalized Newton Iteration Method via Generalized Local Fractional Taylor Series. Advances in Computer Science and its Applications 2012; 1(2): 89-92.